Journal of Difference Equations and Applications,
2001, Vol. 7, pp. 793–810
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Oscillation Results for a Dynamic Equation on a Time Scale*

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(Received in final form 20 August 2000)

First we are concerned with properties of an exponential function for a dynamic equation on a time scale. We completely determine the sign of this exponential function. This then determines when first order linear homogeneous dynamic equations and their adjoints are oscillatory or nonoscillatory. In the last section of this paper we give oscillation criterion for a certain higher order linear dynamic equation on a time scale.

Keywords: Measure chains; Time scales; Oscillation

AMS Subject Classification: 39A10

1. INTRODUCTION

In this paper, we establish some oscillation results for the dynamic equations

$$y^{\Delta} = p(t)y \tag{1}$$

* This paper is dedicated to Calvin Ahlbrandt.

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and

$$y^{\Delta}(\sigma(t)) + p(t)y(t) = 0 \tag{2}$$

on a time scale.

DEFINITION 1 A *measure chain* (*time scale*) is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Assume $\mathbb{T}$ has the topology that it inherits from the real numbers $\mathbb{R}$ with the standard topology.

DEFINITION 2 Let $\mathbb{T}$ be a measure chain and define the *forward jump operator* σ on $\mathbb{T}$ by

$$\sigma(t) := \inf\{s > t : s \in \mathbb{T}\} \in \mathbb{T},$$

for all $t \in \mathbb{T}$. In this definition we put $\sigma(\emptyset) = \sup \mathbb{T}$. The *backward jump operator* ρ on $\mathbb{T}$ is defined by

$$\rho(t) := \sup\{s < t : s \in \mathbb{T}\} \in \mathbb{T},$$

for all $t \in \mathbb{T}$. In this definition we put $\rho(\emptyset) = \inf \mathbb{T}$. If $\sigma(t) > t$, we say t is *right-scattered*, while if $\rho(t) < t$ we say t is *left-scattered*. If $\sigma(t) = t$, we say t is *right-dense*, while if $\rho(t) = t$ we say t is *left-dense*. Finally, the *graininess* function $\mu : \mathbb{T} \mapsto [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

Throughout this paper we make the blanket assumption that $a \leq b$ are points in $\mathbb{T}$.

DEFINITION 3 Define the *interval* $[a, b]$ in $\mathbb{T}$ by

$$[a, b] := \{t \in \mathbb{T} : a \leq t \leq b\}.$$

Other types of intervals are defined similarly. The set $\mathbb{T}^{\kappa}$ is derived from $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^{\kappa} = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$.

We are concerned with calculus on measure chains which was first introduced by Hilger [13] in his dissertation in 1988. Some recent papers concerning dynamic equations on measure chains include Agarwal and Bohner [1, 2], Agarwal, Bohner and Wong [3], Erbe and Hilger [6], Erbe and Peterson [7, 8], Hoffacker [14]. Some preliminary definitions can be found in Kaymakalan, Lakshmikantham and Sivasundaram [15].

Definition 4 Assume $f:\mathbb{T}\mapsto\mathbb{R}$ and let $t\in\mathbb{T}^{\kappa}$, then we define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon>0$, there is a neigborhood U of t such that

$$|[f(\sigma(t))-f(s)]-f^{\Delta}(t)[\sigma(t)-s]|\leq\varepsilon|\sigma(t)-s|,$$

for all $s\in U$. We call $f^{\Delta}(t)$ the *delta derivative* of $f(t)$ at t and it turns out that $^{\Delta}$ is the usual derivative if $\mathbb{T}=\mathbb{R}$ and is the usual forward difference operator Δ if $\mathbb{T}=\mathbb{Z}$.

Some elementary facts that we will use concerning the delta derivative are contained in the following theorem due to Hilger [11].

Theorem 5 *Assume $f:\mathbb{T}\mapsto\mathbb{R}$ is a function and let $t\in\mathbb{T}^{\kappa}$. Then we have the following*:

(1) *If f is differentiable at t, then f is continuous at t.*
(2) *If f is continuous at t and t is right-scattered, then f is differentiable at t with*

$$f^{\Delta}(t)=\frac{f(\sigma(t))-f(t)}{\mu(t)}.$$

(3) *If f is differentiable and t is right-dense, then*

$$f^{\Delta}(t)=\lim_{s\to t}\frac{f(t)-f(s)}{t-s}.$$

(4) *If f is differentiable at t, then*

$$f(\sigma(t))=f(t)+\mu(t)f^{\Delta}(t).$$

Definition 6 A function $F:\mathbb{T}^{\kappa}\mapsto\mathbb{R}$ is called an *antiderivative* of $f:\mathbb{T}\mapsto\mathbb{R}$ provided $F^{\Delta}(t)=f(t)$ holds for all $t\in\mathbb{T}^{\kappa}$. In this case we define the integral of f by

$$\int_a^t f(s)\Delta s=F(t)-F(a)$$

for $t\in\mathbb{T}$.

DEFINITION 7 We say $f : \mathbb{T} \mapsto \mathbb{R}$ is *right-dense continuous* provided f is continuous at each right-dense point $t \in \mathbb{T}$ and whenever $t \in \mathbb{T}$ is left-dense

$$\lim_{s \to t^-} f(s)$$

exists as a finite number.

THEOREM 8 *If f is rd-continuous, then*

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t).$$

For $h > 0$, let $\mathbb{Z}_h$ be the strip in the complex plane

$$\mathbb{Z}_h := \left\{ z \in \mathbb{C} : -\frac{\pi}{h} < Im(z) \leq \frac{\pi}{h} \right\},$$

and for $h = 0$, let $\mathbb{Z}_0 := \mathbb{C}$, the set of complex numbers.

DEFINITION 9 For $h > 0$, we define the *cylinder transformation* $\xi : \mathbb{C}_h \mapsto \mathbb{Z}_h$ by

$$\xi_h(z) = \frac{1}{h}\mathrm{Log}(1 + zh),$$

where Log is the principal logarithm function and $\mathbb{C}_h := \{z \in \mathbb{C} : z \neq -(1/h)\}$. For $h = 0$, we define $\xi_0(z) = z$ for all $z \in \mathbb{Z}_0 = \mathbb{C}$.

Next we define the exponential function.

DEFINITION 10 We say that $p : \mathbb{T}^\kappa \mapsto \mathbb{R}$ is *regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$.*

DEFINITION 11 We say the dynamic equation (1) is *regressive* provided $p : \mathbb{T}^\kappa \mapsto \mathbb{R}$ is regressive and rd-continuous.

DEFINITION 12 If $p : \mathbb{T}^\kappa \mapsto \mathbb{R}$ is regressive and rd-continuous, then we define an *exponential function* (see Hilger [11, 12]) by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau \right)$$

for $t \in \mathbb{T}$, $s \in \mathbb{T}^\kappa$ where $\xi_h(z)$ is the cylinder transformation.

It is well-known that if $\alpha: \mathbb{T}^{\kappa} \mapsto \mathbb{R}$ is rd-continuous and regressive, then the exponential function $e_{\alpha}(t, t_0)$ is for each fixed $t_0 \in \mathbb{T}^{\kappa}$ the unique solution of the initial value problem

$$x^{\Delta} = \alpha(t)x, \ x(t_0) = 1$$

on $\mathbb{T}$.

We will use the following properties of the exponential function $e_p(t, s)$ which are proved in Bohner and Peterson [5, Theorem 3.1].

If p, $q: \mathbb{T}^{\kappa} \mapsto \mathbb{R}$ are regressive and rd-continuous, then the following hold:

- $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$;
- $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
- $e_p(t, s)e_p(s, r) = e_p(t, r)$;
- $e_p(t, s) = (1/e_p(s, t))$;
- $e_p(t, s)e_q(t, s) = e_{p \oplus q}(t, s)$ where $(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t)$, $t \in \mathbb{T}^{\kappa}$.

We will make use of the well-known formula (see Hilger [11])

$$x(\sigma(t)) = x(t) + \mu(t)x^{\Delta}(t).$$

2. EXPONENTIAL FUNCTION AND FIRST ORDER DYNAMIC EQUATIONS

In this section we study the dynamic equation of the form

$$x^{\Delta}(t) = p(t)x(t) \tag{3}$$

where p is rd-continuous and regressive on $\mathbb{T}^{\kappa}$.

It is well-known that

$$x(t) = e_p(t, t_0)x_0$$

is a general solution of Eq. (3).

LEMMA 13 *Suppose* $p: \mathbb{T}^{\kappa} \mapsto \mathbb{R}$ is *rd-continuous and regressive and suppose further there exists a sequence of distinct points* $\{t_n\} \subset \mathbb{T}^{\kappa}$

such that

$$1 + \mu(t_n)p(t_n) < 0$$

$n = 1, 2, \ldots$. *Then* $\lim_{n\to\infty} |t_n| = \infty$.

Proof Assume there exists a sequence of distinct points $\{t_n\} \subset \mathbb{T}^\kappa$ such that

$$1 + \mu(t_n)p(t_n) < 0$$

and

$$\lim_{n\to\infty} t_n = t_0.$$

Since $\mathbb{T}$ is closed, $t_0 \in \mathbb{T}$. Since

$$1 + \mu(t_n)p(t_n) < 0,$$

$\mu(t_n) > 0$ and

$$p(t_n) < -\frac{1}{\mu(t_n)}, \quad n = 1, 2, \ldots . \tag{4}$$

Either there is a subsequence $\{t_{n_k}\}$ such that $t_{n_k} \downarrow t_0$ or $t_{n_k} \uparrow t_0$. Assume $t_{n_k} \downarrow t_0$. Since

$$0 < \mu(t_{n_i}) = \sigma(t_{n_i}) - t_{n_i} \leq t_{n_{i-1}} - t_{n_i},$$

$\lim_{i\to\infty} \mu(t_{n_i}) = 0$.

Assume $t_{n_k} \uparrow t_0$. Since

$$0 < \mu(t_{n_i}) = \sigma(t_{n_i}) - t_{n_i} \leq t_{n_{i+1}} - t_{n_i},$$

$\lim_{i\to\infty} \mu(t_{n_i}) = 0$. So in either case $\lim_{i\to\infty} \mu(t_{n_i}) = 0$ and hence using (4)

$$\lim_{i\to\infty} p(t_i) = -\infty.$$

But this contradicts the fact that p is rd-continuous on $\mathbb{T}$. ∎

THEOREM 14 *Assume* $p : \mathbb{T}^\kappa \mapsto \mathbb{R}$ *is regressive and rd-continuous.*

(1) *If* $1 + \mu(t)p(t) > 0$ *on* $\mathbb{T}^\kappa$, *then* $e_p(t, t_0)$ *is positive on* $\mathbb{T}$.

(2) *If* $1+\mu(\tau)p(\tau)<0$ *for some* $\tau\in\mathbb{T}^\kappa$, *then*

$$e_p(\tau,t_0)e_p(\sigma(\tau),t_0)<0.$$

In this case we say $e_p(t,t_0)$ *has a generalized zero (see Hartman* [10]*) at* $\sigma(\tau)$.

(3) *If* $1+\mu(t)p(t)<0$ *on* $\mathbb{T}^\kappa$, *then* $e_p(t,t_0)$ *changes sign at every point in* $\mathbb{T}$.

(4) *Assume there exist a finite or infinite sequence* $\{t_i\}\subset\mathbb{T}^\kappa$ *and a finite or infinite sequence* $\{s_i\}\subset\mathbb{T}^\kappa$ *with*

$$\cdots<s_2<s_1<t_0\le t_1<t_2<\cdots$$

such that $1+\mu(t_i)p(t_i)<0$ *and* $1+\mu(s_i)p(s_i)<0$ *and* $1+\mu(t)p(t)>0$ *for* $t\in\mathbb{T}^\kappa-[\{s_i\}\cup\{t_i\}]$. *Furthermore if* $\{t_n\}$ *is infinite, then* $\lim_{n\to\infty}t_n=\infty$ *and if* $\{s_n\}$ *is infinite, then* $\lim_{n\to\infty}s_n=-\infty$. *In this case*

$$e_p(t,t_0)>0 \quad on \quad [\sigma(s_1),t_1].$$

If $\{t_n\}$ *is infinite, then*

$$(-1)^i e_p(t,t_0)>0 \quad on \quad [\sigma(t_i),t_{i+1}],\ \ i=1,2,\dots\ .$$

If $\{t_n\}$ *is a finite sequence of N points, then*

$$(-1)^i e_p(t,t_0)>0 \quad on \quad [\sigma(t_i),t_{i+1}],\ \ i=1,2,\dots,N-1$$

and

$$(-1)^N e_p(t,t_0)>0 \quad on \quad [\sigma(t_N),\infty).$$

If $N=0$, *then*

$$e_p(t,t_0)>0 \quad on \quad [\sigma(s_1),\infty).$$

If $\{s_n\}$ *is infinite, then*

$$(-1)^i e_p(t,t_0)>0 \quad on \quad [\sigma(s_{i+1}),s_i],\ \ i=1,2,\dots\ .$$

If card $\{s_n\}=M$, *then*

$$(-1)^i e_p(t,t_0)>0 \quad on \quad [\sigma(s_{i+1}),s_i],\ \ i=1,2,\dots,M-1,$$

and

$$(-1)^M e_p(t,t_0) > 0 \quad on \quad (-\infty, s_M].$$

If $M=0$, *then*

$$e_p(t,t_0) > 0 \quad on \quad (-\infty, t_1].$$

Proof

- Part (1). Assume $1+\mu(t)p(t)>0$ on $\mathbb{T}^\kappa$. If $\mu(t)>0$, then

$$\xi_{\mu(t)}(p(t)) = \frac{1}{\mu(t)} \mathrm{Log}[1+\mu(t)p(t)] \in \mathbb{R}$$

  for all $t \in \mathbb{T}^\kappa$. On the other hand if $\mu(t)=0$, then

$$\xi_{\mu(t)}(p(t)) = p(t) \in \mathbb{R}.$$

  Hence in all cases $\xi_{\mu(t)}(p(t))$ is real for $t \in \mathbb{T}^\kappa$ and hence by definition of the exponential

$$e_p(t,t_0) > 0$$

  on $\mathbb{T}$.
- Part (2). Assume $1+\mu(\tau)p(\tau)<0$ for some $\tau \in \mathbb{T}^\kappa$. It follows that $\mu(\tau)>0$. Since

$$e_p(\sigma(\tau),t_0) = (1+\mu(\tau)p(\tau))e_p(\tau,t_0),$$

  we get the desired result.
- Part (3). Assume $1+\mu(t)p(t)<0$ on $\mathbb{T}^\kappa$. By Theorem 3.2 in [5], $e_p(t,t_0) = \alpha(t,t_0)(-1)^{n_t}$ where

$$\alpha(t,t_0) := \exp\left(\int_{t_0}^{t} \frac{\log|1+\mu(\tau)p(\tau)|}{\mu(\tau)} \Delta\tau\right) > 0$$

  and n_t is defined to be the cardinality of the finite set $[t_0,t)$ if $t \geq t_0$ and n_t is the cardinality of the finite set $[t,t_0)$ if $t<t_0$ so $e_p(t,t_0)$ changes sign at every point in $\mathbb{T}$.
- Part (4). By Lemma 13 the set of points in $\mathbb{T}$ where $1+\mu(t)p(t)<0$ is countable. Let $\{t_n\}$ be the points in $[t_0,\infty)$ such that $1+\mu(t)p(t)<0$.

If $\{t_n\}$ is infinite, then by Lemma 13, $\lim_{n\to\infty} t_n=\infty$. We can assume $t_0 \leq t_1<t_2<\cdots$. Consider the case where there are infinitely many $\{t_i\}\subset\mathbb{T}^\kappa$ with $t_0\leq t_1<t_2<\cdots$ such that $1+\mu(t_i)p(t_i)<0$ and $1+\mu(t)p(t)>0$ for $t\geq t_0$, $t\neq t_i$, $1\leq i<\infty$. We will prove the conclusion of Part (4) for this case by mathematical induction with respect to the intervals $[t_0, t_1]$, $[\sigma(t_1), t_2]$, $[\sigma(t_2), t_3], \ldots$. First we prove that

$$e_p(t, t_0)>0$$

on $[t_0, t_1]$. Since $e_p(t_0, t_0)=1$ and $1+\mu(t)p(t)>0$ on $[t_0, t_1)$, $e_p(t, t_0)>0$ on $[t_0, t_1)$. We now show that $e_p(t_1, t_0)>0$. If $t_1=t_0$, then $e_p(t_1, t_0)=1>0$. Hence we can assume $t_1>t_0$. There are two cases to consider:

Case 1 $\rho(t_1)<t_1$. Then

$$\begin{aligned}
e_p(t_1, t_0) &= e_p(t_1, \rho(t_1))e_p(\rho(t_1), t_0) \\
&= [1+\mu(\rho(t_1))p(\rho(t_1))]e_p(\rho(t_1), \rho(t_1))e_p(\rho(t_1), t_0) \\
&= [1+\mu(\rho(t_1))p(\rho(t_1))]e_p(\rho(t_1), t_0) \\
&> 0.
\end{aligned}$$

Case 2 $\rho(t_1)=t_1$. Since $\rho(t_1)=t_1$, the value of $\int_{t_0}^{t_1} \xi_{\mu(s)}(p(s))\Delta s$ does not depend on the value of the integrand at t_1. Hence $e_p(t_1, t_0)>0$ in this case. Therefore $e_p(t, t_0)>0$ on $[t_0, t_1]$.

Assume $i\geq 0$ and $(-1)^i e_p(t, t_0)>0$ on $[\sigma(t_i), t_{i+1}]$ ($[t_0, t_1]$ *if* $i=0$). It remains to show that

$$(-1)^{i+1}e_p(t, t_0)>0$$

on $[\sigma(t_{i+1}), t_{i+2}]$. Since $1+\mu(t_{i+1})p(t_{i+1})<0$, we have by Part (2) that $\sigma(t_{i+1}) > t_{i+1}$ and

$$e_p(t_{i+1}, t_0)e_p(\sigma(t_{i+1}), t_0)<0.$$

Then

$$(-1)^{i+1}e_p(\sigma(t_{i+1}), t_0)>0. \tag{5}$$

We want to first show that

$$(-1)^{i+1}e_p(t, t_0)>0$$

on $[\sigma(t_{i+1}), t_{i+2})$. If $[\sigma(t_{i+1}), t_{i+2}) = \{\sigma(t_{i+1})\}$, then we are also done by (5). Now assume $(\sigma(t_{i+1}), t_{i+2}) \neq \emptyset$. Let $t \in (\sigma(t_{i+1}), t_{i+2})$. Then

$$\begin{aligned}(-1)^{i+1} e_p(t, t_0) &= (-1)^{i+1} e_p(t, \sigma(t_{i+1})) e_p(\sigma(t_{i+1}), t_0) \\ &= (-1)^{i+1} e^{\int_{\sigma(t_{i+1})}^{t} \xi_{\mu(s)}(p(s))\Delta s} e_p(\sigma(t_{i+1}), t_0) \\ &> 0\end{aligned}$$

by using Eq. (5) and the fact that $1+\mu(t)p(t) > 0$ on $[\sigma(t_{i+1}), t_{i+2})$. It now remains to show that $(-1)^{i+1} e_p(t_{i+2}, t_0) > 0$.

Case 1 $t_{i+1} < \rho(t_{i+2}) < t_{i+2}$. Then

$$\begin{aligned}(-1)^{i+1} e_p(t_{i+2}, t_0) &= (-1)^{i+1} e_p(t_{i+2}, \rho(t_{i+2})) e_p(\rho(t_{i+2}), t_0) \\ &= (-1)^{i+1} [1 + \mu(\rho(t_{i+2})) p(\rho(t_{i+2}))] \\ &\quad e_p(\rho(t_{i+2}), \rho(t_{i+2})) e_p(\rho(t_{i+2}), t_0) \\ &= (-1)^{i+1} [1 + \mu(\rho(t_{i+2})) p(\rho(t_{i+2}))] e_p(\rho(t_{i+2}), t_0) \\ &> 0.\end{aligned}$$

Case 2 $\rho(t_{i+2}) = t_{i+2}$. Since $\rho(t_{i+2}) = t_{i+2}$, the value of $\int_{t_0}^{t_{i+2}} \xi_{\mu(s)}(p(s)) \Delta s$ does not depend on the value of the integrand at t_{i+2} and hence we get the desired result.

The remaining cases are similar and hence are omitted. ∎

The next result follows immediately from Theorem 14.

COROLLARY 15 *If $p: \mathbb{T}^\kappa \mapsto \mathbb{R}$ is rd-continuous and regressive, then $e_p(t, t_0)$ is real-valued and nonzero on $\mathbb{T}$.*

THEOREM 16 *Assume Eq.* (3) *is regressive.*

(1) *If*

$$1 + \mu(t)p(t) > 0 \quad on \quad \mathbb{T}^\kappa,$$

then every nontrivial solution of Eq. (3) *is of one sign on $\mathbb{T}$. If, in addition, $\mathbb{T}$ is unbounded, then we say Eq.* (3) *is nonoscillatory on $\mathbb{T}$ in this case.*

(2) *If*

$$1 + \mu(t)p(t) < 0 \quad on \quad \mathbb{T}^\kappa,$$

then every nontrivial solution of Eq. (3) *changes sign at every point in* $\mathbb{T}$. *If, in addition,* $\mathbb{T}$ *is unbounded, then we say Eq.* (3) *is strongly oscillatory on* $\mathbb{T}$ *in this case.*

(3) *If there exists a strictly increasing sequence* $\{s_i\} \subset \mathbb{T}^\kappa$ *such that*

$$1 + \mu(s_i)p(s_i) < 0 \quad \text{for } i = 1, 2, \ldots,$$

then every nontrivial solution of Eq. (3) *changes sign infinitely often and we say Eq.* (3) *is oscillatory on* $\mathbb{T}$ *in this case.*

(4) *Assume there exists a finite number* N *of points in* $\mathbb{T}^\kappa$ *where* $1+\mu(t)p(t) < 0$, *then every nontrivial solution changes sign exactly* N *times in* $\mathbb{T}$ *and we say Eq.* (3) *is nonoscillatory on* $\mathbb{T}$ *in this case.*

Proof Let $x(t)$ be a nontrivial solution of Eq. (3). Then

$$x(t) = e_p(t, t_0)x_0$$

where $x_0 \neq 0$. The conclusions of this theorem follow from Theorem 14. ∎

In [4] Bohner and Peterson discuss both the nth order linear dynamic equation

$$y^{\Delta^n} + p_1(t)y^{\Delta^{n-1}} + \cdots + p_n(t)y(t) = 0 \tag{6}$$

where

$$1 + \sum_{i=1}^{n}(-\mu(t))^i p_i(t) \neq 0 \quad \text{for all } t \in \mathbb{T}^\kappa$$

and

$$y^{\Delta^n} + q_1(t)(y^{\Delta^{n-1}})^\sigma + \cdots + q_n(t)y^\sigma(t) = 0 \tag{7}$$

where

$$1 + \mu(t)q_1(t) \neq 0.$$

In the first order case we are concerned with the dynamic equations

$$y^\Delta = p(t)y \tag{8}$$

and

$$y^\Delta = -p(t)y^\sigma. \tag{9}$$

Bohner and Peterson [4] call the dynamic equation (9) the adjoint equation of Eq. (8). Since Bohner and Peterson [4] proved that $x(t)$ is a nontrivial solution of Eq. (8) (note a nontrivial solution of Eq. (8) never vanishes.) iff $y(t) = (1/x(t))$ is a nontrivial solution of Eq. (9), we immediately get the following corollary.

COROLLARY 17 *Assume $p : \mathbb{T}^{\kappa} \mapsto \mathbb{R}$ is regressive and rd-continuous.*

(1) *If*

$$1 + \mu(t)p(t) > 0 \quad on \quad \mathbb{T}^{\kappa},$$

then every nontrivial solution of Eq. (9) is of one sign on $\mathbb{T}$. If, in addition, $\mathbb{T}$ is unbounded, then we say Eq. (9) is nonoscillatory on $\mathbb{T}$ in this case.

(2) *If*

$$1 + \mu(t)p(t) < 0 \quad on \quad \mathbb{T}^{\kappa},$$

then every nontrivial solution of Eq. (9) changes sign at every point in $\mathbb{T}$. If, in addition, $\mathbb{T}$ is unbounded, then we say Eq. (9) is strongly oscillatory on $\mathbb{T}$ in this case.

(3) *If there exists a strictly increasing sequence $\{s_i\} \subset \mathbb{T}^{\kappa}$ such that*

$$1 + \mu(s_i)p(s_i) < 0 \quad for\ i = 1, 2, \ldots,$$

then every nontrivial solution of Eq. (9) changes sign infinitely often and we say Eq. (9) is oscillatory on $\mathbb{T}$ in this case.

(4) *Assume there exists a finite number N of points in $\mathbb{T}^{\kappa}$ where $1 + \mu(t)p(t) < 0$, then every nontrivial solution changes sign exactly N times in $\mathbb{T}$ and we say Eq. (9) is nonoscillatory on $\mathbb{T}^{\kappa}$ in this case.*

For higher order linear dynamic equations of the form (6) the relationships between the oscillation of Eq. (7) and its adjoint equation are not known.

3. OSCILLATION AND NONOSCILLATION OF A DYNAMIC EQUATION

In this section we consider the dynamic equation of the form

$$x^{\Delta}(\sigma^{n}(t)) + p(t)x(t) = 0 \tag{10}$$

where p is rd-continuous and regressive on $\mathbb{T}^\kappa$. For Eq. (10) we will only consider measure chains $\mathbb{T}$ where each point in $\mathbb{T}$ is left-scattered and right-scattered. The results in this section are motivated by the work in Chapter 7 of the book [9] due to Győri and Ladas. They study Eq. (10) in the special case where $\mathbb{T}=\mathbb{N}_0$. They show that every solution oscillates if only if an associated characteristic equation has no positive roots. This partially motivates the definition that we give below of oscillation.

One can show that the initial value problem

$$x^\Delta(\sigma^n(t))+p(t)x(t)=0,$$

$$x^{\Delta^i}(t_0)=x_i, \quad 0\le i\le n$$

where $x_i\in\mathbb{R}$, $0\le i\le n$, has a unique solution to the right of $t_0\in\mathbb{T}$.

We look for a solution of Eq. (10) in the form

$$x(t)=e_{\lambda_0}(t,t_0)$$

where $\lambda_0:\mathbb{T}^\kappa\mapsto\mathbb{R}$ is regressive. First note that

$$x^\Delta(t)=\lambda_0(t)e_{\lambda_0}(t,t_0).$$

Then

$$\begin{aligned} x^\Delta(\sigma^n(t)) &= \lambda_0(\sigma^n(t))e_{\lambda_0}(\sigma^n(t),t_0) \\ &= \lambda_0(\sigma^n(t))\prod_{l=0}^{n-1}[1+\mu(\sigma^l(t))\lambda_0(\sigma^l(t))]e_{\lambda_0}(t,t_0) \end{aligned}$$

by using one of the formulas concerning exponential functions given at the end of Section 1. Therefore

$$x^\Delta(\sigma^n(t))+p(t)x(t)=\left\{\lambda_0(\sigma^n(t))\prod_{l=0}^{n-1}[1+\mu(\sigma^l(t))\lambda_0(\sigma^l(t))]+p(t)\right\}e_{\lambda_0}(t,t_0).$$

This leads us to the equation

$$\lambda_0(\sigma^n(t))\prod_{l=0}^{n-1}[1+\mu(\sigma^l(t))\lambda_0(\sigma^l(t))]+p(t)=0 \tag{11}$$

which we call the characteristic equation of Eq. (10). It follows that if $\lambda_0 : \mathbb{T}^\kappa \mapsto \mathbb{R}$ is regressive and $\lambda_0(t)$ satisfies the characteristic equation (11), then

$$x(t) = e_{\lambda_0}(t, t_0)$$

is a solution of Eq. (10) on $\mathbb{T}$.

Example 18 In this example we find exponential solutions of the dynamic equation

$$x^\Delta(\sigma(t)) + px(t) = 0, \tag{12}$$

where p is a nonzero constant, by studying its characteristic equation

$$\lambda_0(\sigma(t))[1 + \mu(t)\lambda_0(t)] + p = 0 \tag{13}$$

when $\mathbb{T} = h\mathbb{Z}$. In this case Eq. (13) is equivalent to

$$\lambda_0(t+h)\lambda_0(t) + \frac{1}{h}\lambda_0(t+h) + \frac{p}{h} = 0.$$

Let $\tilde{\lambda}_0(s) = \lambda_0(hs)$ where $t = hs$. Then we get the Riccati equation (see Kelley and Peterson [16, page 82])

$$\tilde{\lambda}_0(s+1)\tilde{\lambda}_0(s) + \frac{1}{h}\tilde{\lambda}_0(s+1) + \frac{p}{h} = 0.$$

We make the Riccati substitution (see Kelley and Peterson [16, page 82])

$$\tilde{\lambda}_0(s) = \frac{\tilde{x}(s+1)}{\tilde{x}(s)} - \frac{1}{h}. \tag{14}$$

It follows that $\tilde{x}(s)$ is a solution of the second order linear difference equation

$$\tilde{x}(s+2) - \frac{1}{h}\tilde{x}(s+1) + \frac{p}{h}\tilde{x}(s) = 0,$$

which has

$$\tilde{x}(s) = a_1\left(\frac{1 + \sqrt{1 - 4hp}}{2h}\right)^s + a_2\left(\frac{1 - \sqrt{1 - 4hp}}{2h}\right)^s,$$

where a_1 and a_2 are real constants, as a general solution. Therefore by Eq. (14)

$$\tilde{\lambda}_0(s) = \frac{a_1((1+\sqrt{1-4hp})/2h)^{s+1} + a_2((1-\sqrt{1-4hp})/2h)^{s+1}}{a_1((1+\sqrt{1-4hp})/2h)^{s} + a_2((1-\sqrt{1-4hp})/2h)^{s}} - \frac{1}{h},$$

and so

$$\lambda_0(t) = \frac{a_1((1+\sqrt{1-4hp})/2h) + a_2((1-\sqrt{1-4hp})/2h)((1-\sqrt{1-4ph}-2hp)/2hp)^{(t/h)}}{a_1 + a_2((1-\sqrt{1-4hp}-2hp)/2hp)^{(t/h)}} - \frac{1}{h}. \tag{15}$$

In [5] it is shown that if $\mathbb{T} = h\mathbb{Z}$ and $\alpha(t)$ is regressive, then

$$e_\alpha(t,0) = \prod_{j=0}^{(t/h)-1} [1 + h\alpha(jh)].$$

Hence we get for any constants a_1 and a_2 such that $\lambda_0(t)$, given by Eq. (15), is regressive, then

$$e_{\lambda_0}(t,0) = \prod_{j=0}^{(t/h)-1} [1 + h\lambda_0(jh)]$$

is an exponential solution of Eq. (12). In particular if we take $a_1 \neq 0$, $a_2 = 0$ and $a_1 = 0$, $a_2 \neq 0$ respectively, then we get $\lambda_0(t)$ is a constant function and we get the exponential solutions

$$e_{\lambda_0}(t,0) = \left(\frac{1+\sqrt{1-4hp}}{2}\right)^{(t/h)}$$

and

$$e_{\lambda_0}(t,0) = \left(\frac{1-\sqrt{1-4hp}}{2}\right)^{(t/h)}$$

where we assume $p \leq (1/4h)$.

Throughout the remainder of this paper we assume $\sup \mathbb{T} = \infty$. Let $a \in \mathbb{T}$. If there exist a function $\lambda_0(t)$ and $t_0 \in \mathbb{T}$ such that $1 + \mu(t)\lambda_0(t) > 0$ on $[t_0, \infty)$, and $\lambda_0(t)$ satisfies the Eq. (11) on $[t_0, \infty)$, then since $e_{\lambda_0}(t, t_0)$ is a positive solution of Eq. (10) on $[t_0, \infty)$, the Eq. (10) is said to be *nonoscillatory* on $[a, \infty)$. Otherwise we say Eq. (10) is oscillatory on $[a, \infty)$.

THEOREM 19 *Assume there exists a sequence of points* $\{t_k\} \subset [a,\infty)\cap\mathbb{T}$, *with* $t_k \to \infty$ *such that* $p(\rho^{n-i}(t_k)) \geq 0$, $k=1,2,\ldots,i=0,1,2,\ldots,n-1$ *and*

$$p(t_k) \geq \frac{2^n}{\mu(\sigma^n(t_k))}, \quad k=1,2,\ldots.$$

Then Eq. (10) *is oscillatory on* $[a,\infty)\cap\mathbb{T}$.

Proof Assume not, then there exist a function $\lambda_0(t)$ and a $T\in[a,\infty)\cap\mathbb{T}$ such that

$$1+\mu(t)\lambda_0(t) > 0 \tag{16}$$

for $t \geq T$, $t\in[a,\infty)\cap\mathbb{T}$ and $\lambda_0(t)$ satisfies Eq. (11) for $t\geq T$, $t\in[a,\infty)\cap\mathbb{T}$. By (16),

$$\lambda_0(t) > -\frac{1}{\mu(t)}$$

for all $t\geq T$, $t\in[a,\infty)\cap\mathbb{T}$. This implies that

$$\lambda_0(\sigma^i(t)) > -\frac{1}{\mu(\sigma^i(t))} \tag{17}$$

for all $t\geq T$, $t\in[a,\infty)\cap\mathbb{T}$ and $i=0,1,\ldots$. Without loss of generality, assume $t_k > \sigma^{n-1}(T)$ for $k=1,2,\ldots$. Since $\lambda_0(t)$ is a solution of Eq. (11), we get

$$\lambda_0(\sigma^n(t)) = -\frac{p(t)}{\prod_{l=0}^{n-1}[1+\mu(\sigma^l(t))\lambda_0(\sigma^l(t))]}. \tag{18}$$

We claim that $\lambda_0(\sigma^i(t_k)) \leq 0$, $k=1,2,\ldots$ and $i=0,1,\ldots,n-1$. To see this, note that by Eq. (18) with $t=\rho^{n-i}(t_k)$ for $i=0,1,2,\ldots,n-1$ we get

$$\begin{aligned}\lambda_0(\sigma^i(t_k)) &= \lambda_0(\sigma^n(\rho^{n-i}(t_k)))\\ &= -\frac{p(\rho^{n-i}(t_k))}{\prod_{l=0}^{n-1}[1+\mu(\sigma^l(\rho^{n-i}(t_k)))\lambda_0(\sigma^l(\rho^{n-i}(t_k)))]}\\ &\leq 0\end{aligned}$$

for $i=0,1,\ldots,n-1$, $k=1,2,\ldots$. Therefore, we have

$$-\frac{1}{\mu(\sigma^i(t_k))} < \lambda_0(\sigma^i(t_k)) \le 0$$

for $i=0,1,\ldots,n-1$, $k=1, 2,\ldots$. Hence

$$\begin{aligned} 1+\mu(\sigma^i(t_k))\lambda_0(\sigma^i(t_k)) &\le 1+\mu(\sigma^i(t_k))|\lambda_0(\sigma^i(t_k))| \\ &\le 1+\mu(\sigma^i(t_k))\frac{1}{\mu(\sigma^i(t_k))} = 2 \end{aligned}$$

for $i=0,1,\ldots,n-1$, $k=1,2,\ldots$. Since $p(t_k) \ge (2^n/\mu(\sigma^n(t_k))) > 0$,

$$\begin{aligned} \lambda_0(\sigma^n(t_k)) &= -\frac{p(t_k)}{\prod_{l=0}^{n-1}[1+\mu(\sigma^l(t_k))\lambda_0(\sigma^l(t_k))]} \\ &< 0. \end{aligned}$$

Using $1+\mu(\sigma^i(t_k))\lambda_0(\sigma^i(t_k)) \le 2$ for $i=0,1,\ldots,n-1$ and inequality (17),

$$-\frac{1}{\mu(\sigma^n(t_k))} < -\frac{p(t_k)}{\prod_{l=0}^{n-1}[1+\mu(\sigma^l(t_k))\lambda_0(\sigma^l(t_k))]} \le -\frac{p(t_k)}{2^n}.$$

This implies that

$$-\frac{1}{\mu(\sigma^n(t_k))} < -\frac{p(t_k)}{2^n}$$

and consequently

$$p(t_k) < \frac{2^n}{\mu(\sigma^n(t_k))}.$$

But this contradicts the fact that $p(t_k) \ge (2^n/\mu(\sigma^n(t_k)))$. Therefore Eq. (10) is oscillatory. ■

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