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Existence of Bounded Solutions for Second Order Dynamic Equations

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This Paper is dedicated to Professor Lynn H. Erbe

In this paper, we will give sufficient conditions for a nonhomogeneous dynamic self-adjoint equation on a time scale to have a zero tending solution. We also give sufficient conditions that guarantees that for each constant C there is a unique bounded solution on $[a, \infty)$ with $y(a) = C$.

Keywords: Measure chains; Time scales; Self-adjoint; Dynamic equations

INTRODUCTION

For completeness, we introduce the following concepts related to the notion of time scales. We say $\mathbb{T}$ is a time scale, provided, it is a closed subset of the

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real numbers $\mathbb{R}$. We assume throughout that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$. We also assume throughout this paper that $a \in \mathbb{T}$ and $\sup \mathbb{T} = \infty$.

DEFINITION 1 We define the forward jump operator σ, for $t \in \mathbb{T}$, by

$$\sigma(t) := \inf\{\tau > t : \tau \in \mathbb{T}\},$$

and the backward jump operator ρ, for $t \in \mathbb{T}$, $t > \inf \mathbb{T}$, by

$$\rho(t) := \sup\{\tau < t : \tau \in \mathbb{T}\}.$$

If $\sigma(t) > t$, we say t is right-scattered, while if $\rho(t) < t$ we say t is left-scattered. If $\sigma(t) = t$ we say t is right-dense, while if $\rho(t) = t$ we say t is left-dense. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous provided f is continuous at right-dense points in $\mathbb{T}$ and at left-dense points in $\mathbb{T}$, left hand limits exist and are finite. We shall also use the notation $\mu(t) : \sigma(t) - t$ and we call μ the graininess function. Finally, if $f : \mathbb{T} \to \mathbb{R}$ is a function, then we define the function $f^{\sigma} : \mathbb{T} \to \mathbb{R}$ by

$$f^{\sigma}(t) = f(\sigma(t)) \text{ for all } t \in \mathbb{T}.$$

i.e. $f^{\sigma} = f \circ \sigma$. Similarly, $f^{\rho} = f \circ \rho$.

DEFINITION 2 We define the interval in $\mathbb{T}$

$$[a, \infty) := \{t \in \mathbb{T} \text{ such that } t \geq a\}.$$

The notion of a measure chain was introduced by Hilger [10]. Related work on the calculus of measure chains may be found in Refs. [2,3,7–9]. For an introduction to dynamic equations on time scales see Refs. [1,5,6,11].

DEFINITION 3 Assume $x : \mathbb{T} \to \mathbb{R}$ and fix $t \in \mathbb{T}$, then we define $x^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$\|[x(\sigma(t)) - x(s)] - x^{\Delta}(t)[\sigma(t) - s]\| \leq \epsilon|\sigma(t) - s|,$$

for all $s \in U$. We call $x^{\Delta}(t)$ the delta derivative of $x(t)$ at t.

It can be shown that if $x : \mathbb{T} \to \mathbb{R}$ is continuous at $t \in \mathbb{T}$ and t is right-scattered, then

$$x^{\Delta}(t) = \frac{x(\sigma(t)) - x(t)}{\sigma(t) - t}.$$

Note that if $\mathbb{T} = \mathbb{Z}$, the integers, then

$$x^{\Delta}(t) = \Delta x(t) := x(t+1) - x(t).$$

If t is right-dense, then

$$x^{\Delta}(t) = \lim_{s \to t} \frac{x(t) - x(s)}{t - s}.$$

if the limit exists. In particular, if $\mathbb{T} = \mathbb{R}$, the real numbers, then $x^{\Delta}(t) = x'(t)$.

Also an integral $\int_a^b h(t)\Delta t$ can be defined (see Ref. [6]). It turns out if $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b h(t)\Delta t = \int_a^b h(t)\mathrm{d}t$$

is the Riemann integral and if $\mathbb{T} = \mathbb{Z}$ and $a < b$ are integers, then

$$\int_a^b h(t)\Delta t = \sum_{t=a}^{b-1} h(t).$$

MAIN RESULTS

In this section, we will state and prove out main results. We will mainly be concerned with the linear nonhomogeneous dynamic equation with variable coefficients in self-adjoint form:

$$(p(t)y^{\Delta}(t))^{\Delta} + q(t)y^{\sigma}(t) = f(t) \tag{1}$$

where p, q, and f are rd-continuous functions on $\mathbb{T}$ and p is a positive function.

The following result guarantees the existence of a solution of Eq. (1) converging to zero as $t \to \infty$ independent of whether it is oscillatory or nonoscillatory.

THEOREM 4 If

(i) $p(t) > 0, \quad q(t) \geq 0 \quad \text{for all } t \in [a, \infty),$

(ii) $\displaystyle\int_a^\infty \frac{1}{p(\tau)} \Delta\tau < \infty,$

(iii) $\displaystyle\int_a^\infty q(\tau) P^\sigma(\tau) \Delta\tau < \infty, \quad \text{where } P(t) := \int_t^\infty \frac{1}{p(s)} \Delta s,$

(iv) $\displaystyle\int_a^\infty f(\tau) \Delta\tau < \infty,$

then Eq. (1) has a solution which converges to zero as $t \to \infty$.

Proof Let

$$F(t) := \int_t^\infty f(\tau) \Delta\tau$$

and

$$K(t) := \int_t^\infty \frac{F(s)}{p(s)} \Delta s. \tag{2}$$

where K is well defined follows from (ii) and (iv). Therefore $K(t) \to 0$ as $t \to \infty$. Also, $P(t) \to 0$ as $t \to \infty$ and $\int_t^\infty q(\tau) P^\sigma(\tau) \Delta\tau \to 0$ as $t \to \infty$ because of (ii) and (iii), respectively.

By (iii) choose $T \in [a, \infty)$ sufficiently large so that

$$\alpha := \int_T^\infty q(\tau) P^\sigma(\tau) \Delta\tau \in (0, 1). \tag{3}$$

Let X be the Banach Space of all continuous functions $x : [T, \infty) \mapsto \mathbb{R}$ which converge to zero with the norm $\|\cdot\|$ defined by

$$\|y\| = \sup\{|y(t)| : t \in [T, \infty)\},$$

and define the operator A on X by

$$Ay(t) := K(t) + Ly(t), \quad y \in X, \tag{4}$$

where K is defined by Eq. (2) and L is the operator defined by

$$Ly(t) := P(t) \int_T^t q(\tau) y^\sigma(\tau) \Delta\tau + \int_t^\infty q(\tau) P^\sigma(\tau) y^\sigma(\tau) \Delta\tau \tag{5}$$

for all $t \in [T, \infty)$. It is clear that Ay is continuous on $[T, \infty)$. Hence to show $A : X \mapsto X$ it remains to show that $\lim_{t\to\infty} Ay(t) = 0$. Since $K(t) \to 0$ as $t \to \infty$, we need only to show that $Ly(t) \to 0$ as $t \to \infty$. To show this, it suffices to show that if

$$x(t) := P(t)\int_T^t q(\tau)y^{\sigma}(\tau)\Delta\tau,$$

then $\lim_{t\to\infty} x(t) = 0$. To see this, let $\varepsilon > 0$ be given. Choose $t_0 \in \mathbb{T}$ such that $t_0 \geq T$ and

$$|y^{\sigma}(t)| < \varepsilon \quad \text{for all } t > t_0. \tag{6}$$

For this t_0, set

$$\beta = \left|\int_T^{t_0} q(\tau)y^{\sigma}(\tau)\Delta\tau\right| \tag{7}$$

and since $P(t) \to 0$ as $t \to \infty$ we can choose $t_1 \in [t_0, \infty)$ such that

$$P(t)\beta < \varepsilon \tag{8}$$

for all $t \geq t_1$. Then for all $t \geq t_1$,

$$\begin{aligned}
|x(t)| &= \left|P(t)\int_T^t q(\tau)y^{\sigma}(\tau)\Delta\tau\right| = \left|P(t)\int_T^{t_0} q(\tau)y^{\sigma}(\tau)\Delta\tau + P(t)\int_{t_0}^t q(\tau)y^{\sigma}(\tau)\Delta\tau\right| \\
&\leq P(t)\left|\int_T^{t_0} q(\tau)y^{\sigma}(\tau)\Delta\tau\right| + P(t)\int_{t_0}^t q(\tau)|y^{\sigma}(\tau)|\Delta\tau \\
&= P(t)\beta + \int_{t_0}^t q(\tau)P^{\sigma}(\tau)|y^{\sigma}(\tau)|\Delta\tau < \varepsilon + \varepsilon\int_{t_0}^t q(\tau)P^{\sigma}(\tau)\Delta\tau \\
&< \varepsilon(1+\alpha)
\end{aligned}$$

since P is decreasing and we used Eqs. (7), (8), (6), and (3), respectively, (here we used $P(t)\int_{t_0}^t q(\tau)\Delta\tau < \int_{t_0}^t q(\tau)P^{\sigma}(\tau)\Delta\tau$ but this is easy to verify). Hence $\lim_{t\to\infty} x(t) = 0$ and so $A : X \mapsto X$. Next we show that A is a contraction mapping on X. Let $y, z \in X$, $t \geq T$ and consider

$$|Ay(t) - Az(t)| = |Ly(t) - Lz(t)| \leq P(t)\int_T^t q(\tau)|y^\sigma(\tau) - z^\sigma(\tau)|\Delta\tau$$

$$+ \int_t^\infty q(\tau)P^\sigma(\tau)|y^\sigma(\tau) - z^\sigma(\tau)|\Delta\tau$$

$$\leq \int_T^t q(\tau)P^\sigma(\tau)|y^\sigma(\tau) - z^\sigma(\tau)|\Delta\tau + \int_t^\infty q(\tau)P^\sigma(\tau)|y^\sigma(\tau) - z^\sigma(\tau)|\Delta\tau$$

$$= \int_t^\infty q(\tau)P^\sigma(\tau)|y^\sigma(\tau) - z^\sigma(\tau)|\Delta\tau \leq \|y - z\|\int_t^\infty q(\tau)P^\sigma(\tau)\Delta\tau = \alpha\|y - z\|$$

since P is decreasing and by Eq. (3). Therefore

$$\|Ay - Az\| \leq \alpha\|y - z\|$$

for all $y, z \in X$ and so A is a contraction mapping on X. Hence by the Banach Fixed Point Theorem, A has a unique fixed point $y \in X$. Since $y = Ay$, we have $y(t)$ converges to 0 as $t \to \infty$. It remains to show that Eq. (9) is a solution of Eq. (1). Since $y = Ay$,

$$y(t) = K(t) + P(t)\int_T^t q(\tau)y^\sigma(\tau)\Delta\tau + \int_t^\infty q(\tau)P^\sigma(\tau)y^\sigma(\tau)\Delta\tau \qquad (9)$$

for all $t \geq T$. Taking the derivative of both sides we get

$$y^\Delta(t) = K^\Delta(t) + P^\sigma(t)q(t)y^\sigma(t) + P^\Delta(t)\int_T^t q(\tau)y^\sigma(\tau)\Delta\tau + q(t)P^\sigma(t)y^\sigma(t)$$

$$y^\Delta(t) = -\frac{F(t)}{p(t)} - \frac{1}{p(t)}\int_T^t q(\tau)y^\sigma(\tau)\Delta\tau$$

and so

$$p(t)y^\Delta(t) = -F(t) - \int_T^t q(\tau)y^\sigma(\tau)\Delta\tau$$

and therefore

$$(p(t)y^\Delta(t))^\Delta = f(t) - q(t)y^\sigma(t).$$

So we get the desired result. □

DEFINITION 5 A function x is called eventually positive (eventually negative) provided there is a $T \in [a, \infty)$ such that $x(t) \geq 0$ ($x(t) \leq 0$) for all $t \in [T, \infty)$.

COROLLARY 6 Assume that (i)–(iv) of Theorem 4 hold. If f is eventually positive (eventually negative), then Eq. (1) has an eventually positive (eventually negative) solution converging to zero as $t \to \infty$.

Proof Assume f is eventually positive. Without loss of generality we can assume that $f(t) \geq 0$ for all $t \in [T, \infty)$, where T is as in the proof of Theorem 4. Let A be defined by Eqs. (4) and (5) and consider the closed subset $S^+ \subset X$ defined by

$$S^+ = \{y \in X : y(t) \geq 0 \quad \text{for all } t \in [T, \infty)\}. \tag{10}$$

As $K(t) \geq 0$ for all $t \in [T, \infty)$, $AS^+ \subset S^+$. By the same argument as in the proof of Theorem 1, one can show that A is a contraction on S^+. Therefore we get the desired result in the eventual positive case. The eventual negative case is similar. □

REMARK 7 Since $(1/t)^\Delta = -(1/t\sigma(t))$, $\int_1^\infty (1/t\sigma(t))\Delta t = 1$.

Example 8 Assume $\sup \mathbb{T} = \infty$. Consider the dynamic equation

$$(t\sigma(t)x^\Delta(t))^\Delta + \frac{1}{t\sigma(t)}x^\sigma(t) - \frac{1}{t(\sigma(t))^2}$$

on $[1, \infty) \cap \mathrm{T}$. Using Remark 7 we see that the hypotheses of Theorem 1 are satisfied. One of its solutions (use Remark 7) on $\mathbb{T}$ is $x(t) = 1/t$ which converges to zero as $t \to \infty$.

We now consider the analogue of Theorem 4 for the self-adjoint equation

$$[p(t)y^\Delta(t)]^\nabla + q(t)y(t) = f(t) \tag{11}$$

which was introduced by Atici and Guseinov [5]. When considering this equation we assume that $f : \mathbb{T} \to \mathbb{R}$ is continuous, $p : \mathbb{T} \to \mathbb{R}$ is continuous and positive, and $q : \mathbb{T} \to \mathbb{R}$ is continuous. For the definition of the nabla derivative ∇ and the nabla integral $\int_a^b h(t)\nabla t$ see Refs. [4,6].

THEOREM 9 If

(i) $p(t) > 0, q(t) \geq 0 \quad \text{for all } t \in [a, \infty),$

(ii) $\displaystyle\int_a^\infty \frac{1}{p(\tau)} \nabla\tau < \infty, \quad \int_a^\infty \frac{1}{p(\tau)} \Delta\tau < \infty,$

(iii) $\displaystyle\int_a^\infty q^\sigma(\tau) P^\sigma(\tau) \Delta\tau < \infty, \quad \text{where } P(t) := \int_t^\infty \frac{1}{p(s)} \nabla s,$

(iv) $\displaystyle\int_a^\infty f(\tau) \nabla\tau < \infty,$

then Eq. (1) has a solution which converges to zero as $t \to \infty$.

Proof The proof is very similar to the proof of Theorem 4 so we will just indicate how we define some things differently for this case and do the last part of the proof. In this proof F, P, and K are defined by

$$F(t) := \int_t^\infty f(s) \nabla s, \quad P(t) := \int_t^\infty \frac{1}{p(s)} \nabla s, \quad K(t) := \int_t^\infty \frac{F(s)}{p(s)} \Delta s$$

and L and A are defined by

$$Ly(t) := P(t) \int_T^t q(\tau) y(\tau) \nabla\tau + \int_t^\infty P^\sigma(\tau) q^\sigma(\tau) y^\sigma(\tau) \Delta\tau$$

and

$$Ay(t) := Ly(t) + K(t),$$

respectively, where T is sufficiently large. We just show that if y is a fixed point of A, then y is a solution of Eq. (11). To see this consider

$$y(t) = Ay(t) = P(t) \int_T^t q(\tau) y(\tau) \nabla\tau + \int_t^\infty P^\sigma(\tau) q^\sigma(\tau) y^\sigma(\tau) \Delta\tau + K(t).$$

Taking the delta derivative of both sides and using the formula (see Ref. [4] or [6])

$$\left(\int_a^t h(s) \nabla s \right)^\Delta = h(\sigma(t))$$

we get

$$y^{\Delta}(t) = P^{\sigma}(t)q^{\sigma}(t)y^{\sigma}(t) - \frac{1}{p(t)}\int_{T}^{t} q(\tau)y(\tau)\nabla\tau - P^{\sigma}(t)q^{\sigma}(t)y^{\sigma}(t)$$
$$- \frac{1}{p(t)}\int_{t}^{\infty} f(\tau)\nabla\tau.$$

It follows that

$$p(t)y^{\Delta}(t) = -\int_{T}^{t} q(\tau)y(\tau)\nabla\tau - \int_{T}^{\infty} f(\tau)\nabla\tau.$$

Taking the nabla derivative of both sides we get the desired result

$$[p(t)y^{\Delta}(t)]^{\nabla} = -q(t)y(t) + f(t). \qquad \square$$

In the next result we relax the positivity of q and the condition on f guaranteeing the existence of bounded solutions for Eq. (1).

THEOREM 10 Assume there are constants $\underline{\mu}, \bar{\mu}$ such that $0 < \underline{\mu} \leq \mu(t) \leq \bar{\mu}$ for all $t \in [a, \infty)$.

If the conditions

(i) $0 < \alpha \leq p(t) \leq \beta$, for all $t \in [a, \infty)$,

(ii) either

$$q(t) \leq \gamma < 0, \quad \text{for all } t \in [a, \infty), \tag{12}$$

or there is a constant $\delta > 2$ such that

$$\frac{q(t)\mu^2(t)\mu^{\sigma}(t)}{p^{\sigma}(t)\mu(t) + p(t)\mu^{\sigma}(t)} \geq \delta > 2 \quad \text{for all } t \in [a, \infty), \tag{13}$$

(iii) f is bounded on $[a, \infty)$, hold, then for each number C there is a unique bounded solution of Eq. (1) with $y(a) = C$.

Proof Equation (1) can be written in the form:

$$p^{\sigma}(t)\mu(t)y^{\sigma^2}(t) - [p^{\sigma}(t)\mu(t) + p(t)\mu^{\sigma}(t) - q(t)\mu^2(t)\mu^{\sigma}(t)]y^{\sigma}(t)$$
$$+ p(t)\mu^{\sigma}(t)y(t) = \mu^2(t)\mu^{\sigma}(t)f(t)$$

hence

$$y^{\sigma}(t) = \frac{p^{\sigma}(t)\mu(t)y^{\sigma^2}(t) + p(t)\mu^{\sigma}(t)y(t) - \mu^2(t)\mu^{\sigma}(t)f(t)}{p^{\sigma}(t)\mu(t) + p(t)\mu^{\sigma}(t) - q(t)\mu^2(t)\mu^{\sigma}(t)}.$$

Let $t = \rho(s)$, then

$$y(s) = \frac{p(s)\mu^{\rho}(s)y^{\sigma}(s) + p^{\rho}(s)\mu(s)y^{\rho}(s) - \mu^2(\rho(s))\mu(s)f^{\rho}(s)}{p(s)\mu^{\rho}(s) + p^{\rho}(s)\mu(s) - q^{\rho}(s)\mu^2(\rho(s))\mu(s)},$$

for $s \in [\sigma(a), \infty)$.

Consider the Banach Space X of bounded functions on $[a, \infty)$ with the norm $\|y\| = \sup |y(t)|\ t \in [a, \infty)$. Define the operator T by

$$Ty(a) = C,$$

$$Ty(s) = \frac{p(s)\mu^{\rho}(s)y^{\sigma}(s) + p^{\rho}(s)\mu(s)y^{\rho}(s) - \mu^2(\rho(s))\mu(s)f^{\rho}(s)}{p(s)\mu^{\rho}(s) + p^{\rho}(s)\mu(s) - q^{\rho}(s)\mu^2(\rho(s))\mu(s)},$$

for all $s \geq \sigma(a)$.

It is clear that $T : X \mapsto X$. Since the numerator is bounded above and the denominator is bounded away from zero, T is bounded.

Consider the case when condition (12) is satisfied. In this case $p(s)\mu^{\rho}(s) + p^{\rho}(s)\mu(s) - q^{\rho}(s)\mu^2(\rho(s))\mu(s) > 0$. We now show that T is a contraction mapping on X. Let $y, z \in X$, and $s \geq \sigma(a)$. Then

$$|Ty(s) - Tz(s)| = \left|\frac{p(s)\mu^{\rho}(s)(y^{\sigma}(s) - z^{\sigma}(s)) + p^{\rho}(s)\mu(s)(y^{\rho}(s) - z^{\rho}(s))}{p(s)\mu^{\rho}(s) + p^{\rho}(s)\mu(s) - q^{\rho}(s)\mu^2(\rho(s))\mu(s)}\right|$$

$$\leq \frac{p(s)\mu^{\rho}(s)|y^{\sigma}(s) - z^{\sigma}(s)| - p^{\rho}(s)\mu(s)|y^{\rho}(s) - ??z^{\rho}(s)|}{p(s)\mu^{\rho}(s) + p^{\rho}(s)\mu(s) - q^{\rho}(s)\mu^2(\rho(s))\mu(s)}$$

$$\leq \frac{p(s)\mu^{\rho}(s) + p^{\rho}(s)\mu(s)}{p(s)\mu^{\rho}(s) + p^{\rho}(s)\mu(s) - q^{\rho}(s)\mu^2(\rho(s))\mu(s)}\|y - z\|$$

for $s \in [\sigma(a), \infty)$. Consider

$$\frac{p(s)\mu^{\rho}(s) + p^{\rho}(s)\mu(s)}{p(s)\mu^{\rho}(s) + p^{\rho}(s)\mu(s) - q^{\rho}(s)\mu^2(\rho(s))\mu(s)}$$

$$= \frac{1}{1 - \left(\dfrac{q^{\rho}(s)\mu^2(\rho(s))\mu(s)}{p(s)\mu^{\rho}(s) + p^{\rho}(s)\mu(s)}\right)}.$$

But from (i), Eq. (12) and the assumptions on $\mu(t)$, we have

$$\frac{q^{\rho}(s)\mu^2(\rho(s))\mu(s)}{p(s)\mu^{\rho}(s) + p^{\rho}(s)\mu(s)} \leq \frac{\gamma\underline{\mu}^3}{2\beta\bar{\mu}} := \kappa < 0$$

for $s \in [\sigma(a), \infty)$. Therefore

$$|Ty(s) - Tz(s)| \leq \frac{p(s)\mu^\rho(s) + p^\rho(s)\mu(s)}{p(s)\mu^\rho(s) + p^\rho(s)\mu(s) - q^\rho(s)\mu^2(\rho(s))\mu(s)}$$
$$\leq \frac{1}{1-\kappa} < 1$$

for all $s \in [Ty(a), \infty)$. If $s = a$, then

$$|Ty(a) - Tz(a)| = 0 \leq \frac{1}{1-\kappa}\|y - z\|.$$

Therefore

$$|Ty(s) - Tz(s)| < \frac{1}{1-\kappa}\|y - z\|, \quad \text{for all } s \in [a, \infty).$$

This implies that

$$\|Ty - Tz\| \leq \frac{1}{1-\kappa}\|y - z\|$$

and so T is a contraction mapping on X. Hence by the Banach Fixed Point Theorem, T has a unique fixed point y. It follows that

$$y(a) = Ty(a) = C,$$
$$y(s) = Ty(s) = \frac{p(s)\mu^\rho(s)y^\sigma(s) + p^\rho(s)\mu(s)y^\rho(s) - \mu^2(\rho(s))\mu(s)f^\rho(s)}{p(s)\mu^\rho(s) + p^\rho(s)\mu(s) - q^\rho(s)\mu^2(\rho(s))\mu(s)},$$
$$\text{for all } s \geq \sigma(a)$$

is the unique bounded solution of Eq. (1) satisfying $y(a) = C$.

Finally consider the case when Eq. (13) is satisfied. Again we will show that T is a contraction mapping on X. Let $y, z \in X$, and $s \geq \sigma(a)$. Then

$$|Ty(s) - Tz(s)| = \left|\frac{p(s)\mu^\rho(s)(y^\sigma(s) - z^\sigma(s)) + p^\rho(s)\mu(s)(y^\rho(s) - z^\rho(s))}{p(s)\mu^\rho(s) + p^\rho(s)\mu(s) - q^\rho(s)\mu^2(\rho(s))\mu(s)}\right|$$
$$\leq \frac{p(s)\mu^\rho(s)|y^\sigma(s) - z^\sigma(s)| + p^\rho(s)\mu(s)|y^\rho(s) - z^\rho(s)|}{|p(s)\mu^\rho(s) + p^\rho(s)\mu(s) - q^\rho(s)\mu^2(\rho(s))\mu(s)|}$$
$$\leq \frac{p(s)\mu^\rho(s) + p^\rho(s)\mu(s)}{|p(s)\mu^\rho(s) + p^\rho(s)\mu(s) - q^\rho(s)\mu^2(\rho(s))\mu(s)|}\|y - z\|$$

on $[\sigma(a),\infty)$. Consider

$$\frac{p(s)\mu^{\rho}(s)+p^{\rho}(s)\mu(s)}{p(s)\mu^{\rho}(s)|p^{\rho}(s)\mu(s)-q^{\rho}(s)\mu^{2}(\rho(s))\mu(s)}$$

$$=\frac{1}{1-\left(\dfrac{q^{\rho}(s)\mu^{2}(\rho(s))\mu(s)}{p(s)\mu^{\rho}(s)+p^{\rho}(s)\mu(s)}\right)}.$$

But from (ii), Eq. (13) we have

$$2<\delta\leq\frac{q^{\rho}(s)\mu^{2}(\rho(s))\mu(s)}{p(s)\mu^{\rho}(s)+p^{\rho}(s)\mu(s)}$$

for $s\in[\sigma(a),\infty)$. This implies that

$$\frac{1}{1-\delta}\leq\frac{1}{1-\left(\dfrac{q^{\rho}(s)\mu^{2}(\rho(s))\mu(s)}{p(s)\mu^{\rho}(s)+p^{\rho}(s)\mu(s)}\right)}<0$$

for $s\in[\sigma(a),\infty)$. Therefore

$$|Ty(s)-Tz(s)|\leq\left|\frac{p(s)\mu^{\rho}(s)|p^{\rho}(s)\mu(s)}{p(s)\mu^{\rho}(s)|p^{\rho}(s)\mu(s)-q^{\rho}(s)\mu^{2}(\rho(s))\mu(s)}\right|\leq\frac{1}{\delta-1}<1$$

for all $s\in[\sigma(a),\infty)$. If $s=a$, then

$$|Ty(a)-Tz(a)|=0\leq\frac{1}{\delta-1}\|y-z\|.$$

Therefore

$$|Ty(s)-Tz(s)|\leq\frac{1}{\delta-1}\|y-z\|,\quad\text{for all } s\in[a,\infty).$$

Hence

$$\|Ty-Tz\|\leq\frac{1}{\delta-1}\|y-z\|,$$

and so T is a contraction mapping on X. By the Banach Fixed Point Theorem, T has a unique fixed point y. It follows that

$$y(a)=Ty(a)=C,$$

$$y(s) = Ty(s) = \frac{p(s)\mu^{\rho}(s)y^{\sigma}(s) + p^{\rho}(s)\mu(s)y^{\rho}(s) - \mu^2(\rho(s))\mu(s)f^{\rho}(s)}{p(s)\mu^{\rho}(s) + p^{\rho}(s)\mu(s) - q^{\rho}(s)\mu^2(\rho(s))\mu(s)}$$

for all $s \geq \sigma(a)$

is the unique bounded solution of Eq. (1) satisfying $y(a) = C$. □

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