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GENERALIZATION OF MITRINOVIĆ–PEČARIĆ INEQUALITIES ON TIME SCALES

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We prove some new inequalities of Mitrinović–Pečarić inequalities for convex functions on an arbitrary time scale using delta integrals. These inequalities extend and improve some known dynamic inequalities in the literature. The main results will be proved by using Hölder and Jensen inequalities and a simple consequence of Keller's and Poetzsche's chain rules on time scales.

1. Introduction

The following results are known as Opial's inequalities in the literature.

Theorem 1 [20]. *If x is an absolutely continuous function on $[0, h]$ with $x(0) = x(h) = 0$, then*

$$\int_0^h |x(t)x'(t)|dt \leq \frac{h}{4} \int_0^h |x'(t)|^2 dt. \tag{1}$$

Inequality (1) is sharp in the sense that $\frac{h}{4}$ can not be replaced by a smaller constant.

Theorem 2 [7]. *If x is an absolutely continuous function on $[0, h]$ with $x(0) = 0$, then*

$$\int_0^h |x(t)x'(t)|dt \leq \frac{h}{2} \int_0^h |x'(t)|^2 dt,$$

with equality if $x(t) = ct$.

For a comprehensive survey on continuous and discrete Opial-type inequalities, we refer the interested reader to the monograph [2]. The theory of time scales has been initiated by Stefen Hilger in his PhD thesis [15] in order to unify discrete and continuous analysis. A considerable number of dynamic inequalities has been given by many authors who were motivated by some applications; see [1; 3; 4; 5; 6; 10; 13; 14; 16; 18; 21]. The general idea is to prove a result for a dynamic inequality where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which is an arbitrary nonempty closed subset of $\mathbb{R}$. The books [11] and [12] organize and summarize much of time scales calculus. The first dynamic Opial-type inequality (2) was given by Bohner and Kaymakçalan in [9] as follows:

Theorem 3. *Assume that $\mathbb{T}$ is a time scale with $0, h \in \mathbb{T}$. If $x : [0, h]_{\mathbb{T}} \to \mathbb{R}$ is a delta differentiable function with $x(0) = 0$, then*

$$\int_0^h |[x(t) + x^{\sigma}(t)]x^{\Delta}(t)|dt \leq h \int_0^h |x^{\Delta}(t)|^2 dt, \tag{2}$$

with equality if $x(t) = ct$.

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In 1988, Mitrinović and Pečarić [19] discussed several inequalities for the functions that have an integral representation. In this article, motivated by the results obtained in [19], we will state and prove some dynamic inequalities on time scales. The present article is arranged as follows. In Section 2, some basic concepts of the calculus on time scales and useful lemmas are introduced. In Section 3, we state and prove our main results.

2. Preliminaries on time scales

First, we recall some time scales essentials and some universal symbols used in the present article.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. We suppose throughout the article that $\mathbb{T}$ has the topology that it inherits from the standard topology on $\mathbb{R}$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined for any $t \in \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined for any $t \in \mathbb{T}$ by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$, In the previous two definitions, we set $\inf \varnothing = \sup \mathbb{T}$ (i.e., if t is the maximum of $\mathbb{T}$, then $\sigma(t) = t$) and $\sup \varnothing = \inf \mathbb{T}$ (i.e., if t is the minimum of $\mathbb{T}$, then $\rho(t) = t$), where $\varnothing$ is the empty set. A point $t \in \mathbb{T}$ with $\inf \mathbb{T} < t < \sup \mathbb{T}$ is said to be right-scattered if $\sigma(t) > t$, right-dense if $\sigma(t) = t$, left-scattered if $\rho(t) < t$, and left-dense if $\rho(t) = t$. Points that are simultaneously right-dense and left-dense are called dense points. Whereas points that are simultaneously right-scattered and left-scattered are called isolated points. We define the forward graininess function $\mu : \mathbb{T} \to [0, \infty)$ for any $t \in \mathbb{T}$ by $\mu(t) := \sigma(t) - t$. We introduce the set $\mathbb{T}^\kappa$ as follows: If $\mathbb{T}$ has a left-scattered maximum t_1, then $\mathbb{T}^\kappa = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. The interval $[a, b]$ in $\mathbb{T}$ is defined by $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \le t \le b\}$. Let $f : \mathbb{T} \to \mathbb{R}$ be a function. Then $f^\sigma : \mathbb{T} \to \mathbb{R}$ is defined by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$.

Suppose $f : \mathbb{T} \to \mathbb{R}$ is a function and $t \in \mathbb{T}^\kappa$. Then we say that $f^\Delta(t) \in \mathbb{R}$ is the delta derivative of f at t if for any $\varepsilon > 0$ there exists a neighborhood U of t such that, for all $s \in U$, we have

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \le \varepsilon|\sigma(t) - s|.$$

Furthermore, f is said to be delta differentiable on $\mathbb{T}^\kappa$ if it is delta differentiable at each $t \in \mathbb{T}^\kappa$. If $f, g : \mathbb{T} \to \mathbb{R}$ are delta differentiable functions at $t \in \mathbb{T}^\kappa$, then

$$\text{(i)} \qquad (fg)^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g^\sigma(t),$$

$$\text{(ii)} \quad \left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g^\sigma(t)}, \qquad g(t)g(\sigma(t)) \neq 0.$$

A function $g : \mathbb{T} \to \mathbb{R}$ is called right-dense continuous (rd-continuous) if g is continuous at the right-dense points in $\mathbb{T}$ and its left-sided limits exist at all left-dense points in $\mathbb{T}$. A function $F : \mathbb{T} \to \mathbb{R}$ is said to be a delta antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^\kappa$. In this case, the definite delta integral of f is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a) \qquad \text{for all } a, b \in \mathbb{T}.$$

In the following, we present the basic theorems that will be needed in the proof of our main results.

Theorem 4 (chain rule on time scales [11]). *Assume $g : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}^\kappa$, and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then there exists $c \in [t, \sigma(t)]$ with*

$$(f \circ g)^\Delta(t) = f'(g(c))g^\Delta(t). \tag{3}$$

Theorem 5 (chain rule on time scales [11]). *Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable and the formula*

$$(f \circ g)^{\Delta}(t) = \left\{ \int_0^1 [f'(g(t) + \mu(t)hg^{\Delta}(t))]dh \right\} g^{\Delta}(t) \tag{4}$$

holds.

Theorem 6 (dynamic Hölder's inequality [8]). *Let $a, b \in \mathbb{T}$ and $f, g \in C_{\mathrm{rd}}([a, b]_{\mathbb{T}}, [0, \infty))$. If $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\int_a^b f(t)g(t)\Delta t \leq \left[\int_a^b f^p(t)\Delta t \right]^{\frac{1}{p}} \left[\int_a^b g^q(t)\Delta t \right]^{\frac{1}{q}}. \tag{5}$$

Theorem 7 (dynamic Jensen inequality [4]). *Let $u \in C_{\mathrm{rd}}([a, b]_{\mathbb{T}}, (c, d))$ and $v \in C_{\mathrm{rd}}([a, b]_{\mathbb{T}}, \mathbb{R})$ such that*

$$\int_a^b |v(\tau)|\Delta\tau > 0.$$

Moreover, suppose $F \in C((c, d), \mathbb{R})$ is a convex function. Then

$$F\left(\frac{\int_a^b u(\tau)|v(\tau)|\Delta\tau}{\int_a^b |v(\tau)|\Delta\tau} \right) \leq \frac{\int_a^b F(u(\tau))|v(\tau)|\Delta\tau}{\int_a^b |v(\tau)|\Delta\tau}. \tag{6}$$

3. Main results

In this section, we will generalize Mitrinović–Pečarić inequalities which have integral representations. We say that the function x belongs to the class $U(y, K)$ if it can be represented in the form

$$x(t) = \int_\alpha^\tau K(t, s)y(s)\Delta s, \quad t \in [\alpha, \tau]_{\mathbb{T}}, \tag{7}$$

where $y \in C_{\mathrm{rd}}([\alpha, \tau]_{\mathbb{T}}, \mathbb{R})$ and $K(t, s)$ is an arbitrary nonnegative kernel defined on $[\alpha, \tau]_{\mathbb{T}} \times [\alpha, \tau]_{\mathbb{T}}$ such that $x(t) > 0$ if $y(t) > 0$ for $t \in [\alpha, \tau]_{\mathbb{T}}$.

Theorem 8. *Let $\mathbb{T}$ be a time scale with $\alpha, \tau \in \mathbb{T}$. For $i = 1, 2$, let $x_i \in U(y_i, K)$, where $y_2(t) > 0$, $t \in [\alpha, \tau]_{\mathbb{T}}$. Further, let $p \in C_{\mathrm{rd}}([\alpha, \tau]_{\mathbb{T}}, [0, \infty))$, and let f be convex and increasing on $[0, \infty)$. Then the inequality*

$$\int_\alpha^\tau p(t) f\left(\left| \frac{x_1(t)}{x_2(t)} \right| \right) \Delta t \leq \int_\alpha^\tau \phi(t) f\left(\left| \frac{y_1(t)}{y_2(t)} \right| \right) \Delta t \tag{8}$$

holds, where

$$\phi(t) = y_2(t) \int_\alpha^\tau \frac{p(s)K(t, s)}{x_2(s)} \Delta s. \tag{9}$$

Proof. Since $x_1 \in U(y_1, K)$, from (7) we have

$$
\begin{aligned}
\int_\alpha^\tau p(t) f\left(\left|\frac{x_1(t)}{x_2(t)}\right|\right)\Delta t &= \int_\alpha^\tau p(t) f\left(\left|\int_\alpha^\tau \frac{K(t,s)y_1(s)}{x_2(t)}\Delta s\right|\right)\Delta t \\
&= \int_\alpha^\tau p(t) f\left(\left|\int_\alpha^\tau \frac{K(t,s)y_2(s)}{x_2(t)}\frac{y_1(s)}{y_2(s)}\Delta s\right|\right)\Delta t \\
&\le \int_\alpha^\tau p(t) f\left(\int_\alpha^\tau \frac{K(t,s)y_2(s)}{x_2(t)}\frac{|y_1(s)|}{y_2(s)}\Delta s\right)\Delta t.
\end{aligned} \tag{10}
$$

Applying dynamic Jensen's inequality (6) on (10), we get

$$
\begin{aligned}
\int_\alpha^\tau p(t) f\left(\left|\frac{x_1(t)}{x_2(t)}\right|\right)\Delta t &\le \int_\alpha^\tau p(t) \int_\alpha^\tau \frac{K(t,s)y_2(s)}{x_2(t)} f\left(\frac{|y_1(s)|}{y_2(s)}\right)\Delta s \Delta t \\
&= \int_\alpha^\tau f\left(\frac{|y_1(s)|}{y_2(s)}\right) y_2(s)\left(\int_\alpha^\tau \frac{p(t)K(t,s)}{x_2(t)}\Delta t\right)\Delta s \\
&= \int_\alpha^\tau \phi(t) f\left(\frac{|y_1(s)|}{y_2(t)}\right)\Delta t,
\end{aligned}
$$

which is the desired inequality (8), where ϕ is defined as in (9). □

Note that when $\mathbb{T} = \mathbb{R}$, the inequality (8) in Theorem 8 reduces to the inequality in Theorem 2.22.1 of [2]. Moreover, Theorem 8 can be generalized to convex functions of several variables similarly. In the continuous case, inequality (11) in Theorem 9 turns out to be the inequality in Theorem 2.22.2 in [2].

Theorem 9. *Let $\mathbb{T}$ be a time scale with $\alpha, \tau \in \mathbb{T}$. For $i = 1, 2, 3$, let $x_i \in U(y_i, K)$, where $y_2(t) > 0$, $t \in [\alpha, \tau]_\mathbb{T}$. Further, let $p \in C_{\mathrm{rd}}([\alpha, \tau]_\mathbb{T}, [0, \infty))$, and $f(\cdot, \cdot)$ be convex and increasing on $[0, \infty) \times [0, \infty]$. Then we have the inequality*

$$
\int_\alpha^\tau p(t) f\left(\left|\frac{x_1(t)}{x_2(t)}\right|, \left|\frac{x_3(t)}{x_2(t)}\right|\right)\Delta t \le \int_\alpha^\tau \phi(t) f\left(\frac{|y_1(t)|}{y_2(t)}, \frac{|y_3(t)|}{y_2(t)}\right)\Delta t, \tag{11}
$$

where $\phi(t)$ is given as in (9).

Now, let $x(t) \in U(y, K)$, where $K(t, s) = 0$ for $s > t$. We say that such functions belong to the class $U_1(y, K)$. It is clear that in this case, (7) reduces to

$$
x(t) = \int_\alpha^t K(t,s)y(s)\Delta s. \tag{12}
$$

Theorem 10. *Let $\mathbb{T}$ be a time scale with $\alpha, \tau \in \mathbb{T}$. For $i = 1, 2$, let $x_i(t) \in U_1(y_i, K)$, where $y_2(t) > 0$, $t \in [\alpha, \tau]_\mathbb{T}$. Furthermore, let f and g be convex and increasing on $[0, \infty)$ such that $f(0) = 0$. If $f(t)$ is also differentiable and* $\max K(t, s) = M$, *then we obtain the inequality*

$$
M\int_\alpha^\tau y_2(t) g\left(\frac{|y_1(t)|}{y_2(t)}\right) f'\left(x_2(t) g\left(\left|\frac{x_1(t)}{x_2(t)}\right|\right)\right)\Delta t \le f\left(M\int_\alpha^\tau y_2(t) g\left(\frac{|y_1(t)|}{y_2(t)}\right)\Delta t\right). \tag{13}
$$

Proof. Since f' is a nondecreasing function and from (12) for $x_1(t)$, we have

$$
\begin{aligned}
(14)\quad & M\int_\alpha^\tau y_2(t)g\left(\frac{|y_1(t)|}{y_2(t)}\right)f'\left(x_2(t)g\left(\left|\frac{x_1(t)}{x_2(t)}\right|\right)\right)\Delta t \\
&= M\int_\alpha^\tau y_2(t)g\left(\frac{|y_1(t)|}{y_2(t)}\right)f'\left(x_2(t)g\left(\left|\int_\alpha^t \frac{K(t,s)y_1(s)}{x_2(t)}\Delta s\right|\right)\right)\Delta t \\
&= M\int_\alpha^\tau y_2(t)g\left(\frac{|y_1(t)|}{y_2(t)}\right)f'\left(x_2(t)g\left(\left|\int_\alpha^t \frac{K(t,s)y_2(s)}{x_2(t)}\frac{y_1(s)}{y_2(s)}\Delta s\right|\right)\right)\Delta t \\
&\le M\int_\alpha^\tau y_2(t)g\left(\frac{|y_1(t)|}{y_2(t)}\right)f'\left(x_2(t)g\left(\int_\alpha^t \frac{K(t,s)y_2(s)}{x_2(t)}\frac{|y_1(s)|}{y_2(s)}\Delta s\right)\right)\Delta t
\end{aligned}
$$

Applying the dynamic Jensen's inequality (6) for the convex function g on (14), we have

$$
\begin{aligned}
& M\int_\alpha^\tau y_2(t)g\left(\frac{|y_1(t)|}{y_2(t)}\right)f'\left(x_2(t)g\left(\left|\frac{x_1(t)}{x_2(t)}\right|\right)\right)\Delta t \\
&\le M\int_\alpha^\tau y_2(t)g\left(\frac{|y_1(t)|}{y_2(t)}\right)f'\left(\int_\alpha^t K(t,s)y_2(s)g\left(\frac{|y_1(s)|}{y_2(s)}\right)\Delta s\right)\Delta t.
\end{aligned}
$$

Since $\max K(t,s)=M$, and from the Keller's chain rule on time scales (3), for $c\in[t,\sigma(t)]$, we get

$$
\begin{aligned}
M\int_\alpha^\tau y_2(t)g\left(\frac{|y_1(t)|}{y_2(t)}\right)f'\left(x_2(t)g\left(\left|\frac{x_1(t)}{x_2(t)}\right|\right)\right)\Delta t &\le M\int_\alpha^\tau y_2(t)g\left(\frac{|y_1(t)|}{y_2(t)}\right)f'\left(\int_\alpha^t My_2(s)g\left(\frac{|y_1(s)|}{y_2(s)}\right)\Delta s\right)\Delta t \\
&\le M\int_\alpha^\tau y_2(t)g\left(\frac{|y_1(t)|}{y_2(t)}\right)f'\left(\int_\alpha^c My_2(s)g\left(\frac{|y_1(s)|}{y_2(s)}\right)\Delta s\right)\Delta t \\
&= \int_\alpha^\tau \left(f\left(\int_\alpha^t My_2(s)g\left(\frac{|y_1(s)|}{y_2(s)}\right)\Delta s\right)\right)^\Delta \Delta t \\
&= f\left(M\int_\alpha^\tau y_2(t)g\left(\frac{|y_1(t)|}{y_2(t)}\right)\Delta t\right),
\end{aligned}
$$

which is the desired inequality (13). □

In the continuous case, the inequality (13) in Theorem 10 becomes the inequality in Theorem 2.22.2 in [2]. If we let $g(t)=t$, $M=1$, $x_1=x$, and $y_1=x^\Delta$, then we obtain

$$
\int_\alpha^\tau f'(|x(t)|)|x^\Delta(t)|\Delta t \le f\left(\int_\alpha^\tau |x^\Delta(t)|\Delta t\right),
$$

that is, the inequality in Theorem 2.17.1 in the continuous case in [2]. For example, $M=1$ if $\lambda=1$, where

$$
K(t,s)=K_\lambda(t,s)=\begin{cases}\frac{(t-s)^{\lambda-1}}{\Gamma(\lambda)}, & s\le t,\\ 0, & s>t.\end{cases}
$$

If $\lambda=1$ and x is delta differentiable, then we have $y=x^\Delta$.

Theorem 11. *Let $\mathbb{T}$ be a time scale with $\alpha,\tau\in\mathbb{T}$, and f be differentiable, convex, and increasing on $[0,\infty)$ such that $f(0)=0$. Furthermore, let $x(t)\in U_1(y,K)$, where $\left(\int_\alpha^t K^\eta(t,s)\Delta s\right)^{1/\eta}\le M$, and*

$\frac{1}{\eta}+\frac{1}{\nu}=1$. *Then we have the inequality*

$$\int_{\alpha}^{\tau}|x^{\sigma}(t)|^{1-\nu}f'(|x(t)|)|y(t)|^{\nu}\Delta t\leq\frac{\nu}{M^{\nu}}f\bigg(M\bigg(\int_{\alpha}^{\tau}|y(t)|^{\nu}\Delta t\bigg)^{\frac{1}{\nu}}\bigg).\tag{15}$$

Proof. Since $x(t)=\int_{\alpha}^{t}K(t,s)y(s)\Delta s$, we obtain

$$|x(t)|=\bigg|\int_{\alpha}^{t}K(t,s)y(s)\Delta s\bigg|\leq\int_{\alpha}^{t}K(t,s)|y(s)|\Delta s.$$

By applying the dynamic Hölder's inequality (5), with indices η and ν, we have

$$|x(t)|\leq\bigg(\int_{\alpha}^{t}K^{\eta}(t,s)\Delta s\bigg)^{\frac{1}{\eta}}\bigg(\int_{\alpha}^{t}|y(t)|^{\nu}\Delta s\bigg)^{\frac{1}{\nu}}\leq M\bigg(\int_{\alpha}^{t}|y(t)|^{\nu}\Delta s\bigg)^{\frac{1}{\nu}}.$$

Now, let $z(t)=\int_{\alpha}^{t}|y(s)|^{\nu}\Delta s$. Then $z^{\Delta}(t)=|y(t)|^{\nu}$, and $|x(t)|\leq M(z(t))^{1/\nu}$. By the chain rule on time scales (4), we have

$$(Mz^{\frac{1}{\nu}}(t))^{\Delta}\geq\frac{Mz^{\Delta}(t)}{\nu(z^{\sigma}(t))^{1-\frac{1}{\nu}}},\tag{16}$$

where we use $z+\mu hz^{\Delta}\leq z^{\sigma}$. Note that using the chain rule (4) once again, we can show that

$$(f(Mz^{\frac{1}{\nu}}(t)))^{\Delta}\geq\frac{M}{\nu}\frac{z^{\Delta}(t)}{(z^{\sigma}(t))^{1-\frac{1}{\nu}}}f'(Mz^{\frac{1}{\nu}}).\tag{17}$$

Then from (16) and (17) we have

$$\begin{aligned}&\int_{\alpha}^{\tau}|x^{\sigma}(t)|^{1-\nu}f'(|x(t)|)|y(t)|^{\nu}\Delta t\\&\quad\leq\int_{\alpha}^{\tau}(M(z^{\sigma}(t))^{\frac{1}{\nu}})^{1-\nu}f'(M(z(t))^{\frac{1}{\nu}})z^{\Delta}(t)\Delta t=\int_{\alpha}^{\tau}M^{1-\nu}(z^{\sigma}(t))^{\frac{1}{\nu}-1}f'(M(z(t))^{\frac{1}{\nu}})z^{\Delta}(t)\Delta t\\&\quad\leq\frac{\nu}{M^{\nu}}\int_{\alpha}^{\tau}(f(M(z(t))^{\frac{1}{\nu}}))^{\Delta}\Delta t=\frac{\nu}{M^{\nu}}f(M(z(\tau))^{\frac{1}{\nu}})=\frac{\nu}{M^{\nu}}f\bigg(M\bigg(\int_{\alpha}^{\tau}|y(t)|^{\nu}\Delta t\bigg)^{\frac{1}{\nu}}\bigg),\end{aligned}$$

which is the desired inequality (15). □

Note that when $\mathbb{T}=\mathbb{R}$, the inequality (15) reduces to the inequality in Theorem 2.22.4 in [2] . Here, we do not assume that $f(t^{1/\nu})$ is convex in Theorem 11 as in the continuous case.

Analogous to $U(y,K)$ we can define the class $\bar{U}(y,K)$, where $x(t)\in\bar{U}(y,K)$ has the representation

$$x(t)=\int_{\tau}^{\beta}K(t,s)y(s)\Delta s,\qquad t\in[\tau,\beta]_{\mathbb{T}}\tag{18}$$

where y is an rd-continuous function on $[\tau,\beta]_{\mathbb{T}}$, and $K(t,s)$ is an arbitrary nonnegative kernel defined on $[\tau,\beta]_{\mathbb{T}}\times[\tau,\beta]_{\mathbb{T}}$ such that $x(t)>0$ if $y(t)>0$, $t\in[\tau,\beta]_{\mathbb{T}}$.

It is obvious that the results for the functions belonging to the class $\bar{U}(y,K)$ can be obtained similarly by replacing $[\alpha,\tau]_{\mathbb{T}}$ by $[\tau,\beta]_{\mathbb{T}}$.

Theorem 12. *Let $\mathbb{T}$ be a time scale with $\tau, \beta \in \mathbb{T}$. For $i = 1, 2$, let $x_i \in \bar{U}(y_i, K)$, where $y_2(t) > 0$, $t \in [\tau, \beta]_{\mathbb{T}}$. Further, let $p \in C_{\rm rd}([\alpha, \tau]_{\mathbb{T}}, [0, \infty))$, and let f be convex and increasing on $[0, \infty)$. Then,*

$$\int_\tau^\beta p(t) f\left(\left|\frac{x_1(t)}{x_2(t)}\right|\right)\Delta t \leq \int_\tau^\beta \phi(t) f\left(\frac{|y_1(t)|}{y_2(t)}\right)\Delta t,$$

where

$$\phi(t) = y_2(t)\int_\tau^\beta \frac{p(s)K(s,t)}{x_2(s)}\Delta s. \tag{19}$$

Theorem 13. *Let $\mathbb{T}$ be a time scale with $\tau, \beta \in \mathbb{T}$. For $i = 1, 2, 3$, let $x_i \in \bar{U}(y_i, K)$, where $y_2(t) > 0$, $t \in [\tau, \beta]_{\mathbb{T}}$. Furthermore, let $p \in C_{\rm rd}([\alpha, \tau]_{\mathbb{T}}, [0, \infty))$, and let $f(\cdot, \cdot)$ be convex and increasing on $[0, \infty) \times [0, \infty)$. Then,*

$$\int_\tau^\beta p(t) f\left(\left|\frac{x_1(t)}{x_2(t)}\right|, \left|\frac{x_3(t)}{x_2(t)}\right|\right)\Delta t \leq \int_\tau^\beta \phi(t) f\left(\frac{|y_1(t)|}{y_2(t)}, \frac{|y_3(t)|}{y_2(t)}\right)\Delta t,$$

where $\phi(t)$ is given as in (19).

Now, let $x(t) \in \bar{U}(y, K)$, where $K(t, s) = 0$ for $s > t$. We say that such functions belong to the class $\bar{U}_1(y, K)$. It is clear that, in this case, (18) reduces to

$$x(t) = \int_t^\beta K(t,s) y(s)\Delta s.$$

It is clear that similar results for the functions belonging to the class $\bar{U}_1(y, K)$ can be obtained by replacing $[\alpha, \tau]_{\mathbb{T}}$ by $[\tau, \beta]_{\mathbb{T}}$.

Theorem 14. *Let $\mathbb{T}$ be a time scale with $\tau, \beta \in \mathbb{T}$. For $i = 1, 2$, let $x_i(t) \in \bar{U}_1(y_i, K)$, where $y_2(t) > 0$, $t \in [\tau, \beta]_{\mathbb{T}}$. Furthermore, let f and g be convex and increasing on $[0, \infty)$, and let $f(0) = 0$. If f is also differentiable and* $\max K(t, s) = M$, *then, the following inequality holds*

$$M\int_\tau^\beta y_2(t) g\left(\frac{|y_1(t)|}{y_2(t)}\right) f'\left(x_2(t) g\left(\left|\frac{x_1(t)}{x_2(t)}\right|\right)\right)\Delta t \leq f\left(M\int_\tau^\beta y_2(t) g\left(\frac{|y_1(t)|}{y_2(t)}\right)\Delta t\right).$$

Theorem 15. *Let $\mathbb{T}$ be a time scale with $\tau, \beta \in \mathbb{T}$, and let f be differentiable, convex and increasing on $[0, \infty)$ such that $f(0) = 0$. Further, let $x(t) \in \bar{U}_1(y, K)$, where $\left(\int_\beta^t K^\eta(t,s)\Delta s\right)^{1/\eta} \leq M$, and $\frac{1}{\eta} + \frac{1}{\nu} = 1$. Then we have*

$$\int_\tau^\beta |x^\sigma(t)|^{1-\nu} f'(|x(t)|)|y(t)|^\nu \Delta t \leq \frac{\nu}{M^\nu} f\left(M\left(\int_\tau^\beta |y(t)|^\nu \Delta t\right)^{\frac{1}{\nu}}\right).$$

Motivated by [17], we now remove the absolute values in (8) and obtain the following theorem whose proof is similar to the proof of Theorem 8.

Theorem 16. *Let $u_i \in U(v, K)$, $i = 1, 2$, and $r \in C_{\rm rd}([a, b]_{\mathbb{T}}, \mathbb{R}^+)$. Also let f be a convex function on a compact interval $I \subseteq \mathbb{R}^+$. Further, let $v_1, v_2 : [a, b]_{\mathbb{T}} \mapsto I$ such that $u_1/u_2, v_1/v_2 \in I$, and v_1/v_2 is nonconstant. Then,*

$$\int_a^b r(t) f\left(\frac{u_1(t)}{u_2(t)}\right)\Delta t \leq \int_a^b q(t) f\left(\frac{v_1(t)}{v_2(t)}\right)\Delta t, \tag{20}$$

where

$$q(t) = v_2(t)\int_a^b \frac{r(s)K(s,t)}{u_2(s)}\Delta s. \tag{21}$$

Lemma 17. *Let $h \in C^2(I)$, and I be a compact interval such that*

$$m \le h''(t) \le M \quad \textit{for all } t \in I,$$

and define f_1, f_2 by

$$f_1(t) = \frac{M}{2}t^2 - h(t),$$
$$f_2(t) = h(t) - \frac{m}{2}t^2.$$

Then f_1 and f_2 are convex on I.

Remark 18. If f is strictly convex on I and v_1/v_2 is nonconstant, then the inequality in (20) is strict.

Theorem 19. *Let $h \in C^2(I)$, I be a compact interval of $\mathbb{R}^+$, let $u_i \in U(v,K)$, $i = 1,2$, and let $r \in C_{\mathrm{rd}}([a,b]_{\mathbb{T}}, \mathbb{R}^+)$. Furthermore, assume that $v_1, v_2 : [a,b]_{\mathbb{T}} \mapsto I$ such that $u_1/u_2, v_1/v_2 \in I$, and v_1/v_2 is nonconstant. Then there exists $\xi \in I$ such that*

$$\frac{\int_a^b\left[q(t)h\left(\frac{v_1(t)}{v_2(t)}\right) - r(t)h\left(\frac{u_1(t)}{u_2(t)}\right)\right]\Delta t}{\int_a^b\left[q(t)\left(\frac{v_1(t)}{v_2(t)}\right)^2 - r(t)\left(\frac{u_1(t)}{u_2(t)}\right)^2\right]\Delta t} = \frac{h''(\xi)}{2}, \tag{22}$$

where q is given as in (21).

Proof. Since $h \in C^2(I)$ and I is compact, f_1 and f_2 described in Lemma 17 are convex on I. Suppose that $m = \min h''$ and $M = \max h''$. By using Theorem 16 for f_1, we get

$$\int_a^b r(t) f_1\left(\frac{u_1(t)}{u_2(t)}\right)\Delta t \le \int_a^b q(t) f_1\left(\frac{v_1(t)}{v_2(t)}\right)\Delta t.$$

This implies that

$$\int_a^b r(t)\left[\frac{M}{2}\left(\frac{u_1(t)}{u_2(t)}\right)^2 - h\left(\frac{u_1(t)}{u_2(t)}\right)\right]\Delta t \le \int_a^b q(t)\left[\frac{M}{2}\left(\frac{v_1(t)}{v_2(t)}\right)^2 - h\left(\frac{v_1(t)}{v_2(t)}\right)\right]\Delta t$$

or

$$\frac{M}{2}\int_a^b\left[q(t)\left(\frac{v_1(t)}{v_2(t)}\right)^2 - r(t)\left(\frac{u_1(t)}{u_2(t)}\right)^2\right]\Delta t \ge \int_a^b\left[q(t)h\left(\frac{v_1(t)}{v_2(t)}\right) - r(t)h\left(\frac{u_1(t)}{u_2(t)}\right)\right]\Delta t.$$

Note that the integral on the left of the above inequality is positive because of Remark 18. Therefore, we have

$$\frac{\int_a^b\left[q(t)h\left(\frac{v_1(t)}{v_2(t)}\right) - r(t)h\left(\frac{u_1(t)}{u_2(t)}\right)\right]\Delta t}{\int_a^b\left[q(t)\left(\frac{v_1(t)}{v_2(t)}\right)^2 - r(t)\left(\frac{u_1(t)}{u_2(t)}\right)^2\right]\Delta t} \le \frac{M}{2}. \tag{23}$$

Similarly, we can now apply Theorem 16 for f_2 and obtain

$$\frac{\int_a^b\left[q(t)h\left(\frac{v_1(t)}{v_2(t)}\right)-r(t)h\left(\frac{u_1(t)}{u_2(t)}\right)\right]\Delta t}{\int_a^b\left[q(t)\left(\frac{v_1(t)}{v_2(t)}\right)^2-r(t)\left(\frac{u_1(t)}{u_2(t)}\right)^2\right]\Delta t}\geq\frac{m}{2}. \tag{24}$$

Combining (23) and (24) gives us

$$m\leq 2\frac{\int_a^b\left[q(t)h\left(\frac{v_1(t)}{v_2(t)}\right)-r(t)h\left(\frac{u_1(t)}{u_2(t)}\right)\right]\Delta t}{\int_a^b\left[q(t)\left(\frac{v_1(t)}{v_2(t)}\right)^2-r(t)\left(\frac{u_1(t)}{u_2(t)}\right)^2\right]\Delta t}\leq M.$$

Finally, by using Lemma 17 there exists $\xi\in I$ such that (22) holds. □

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