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On nonoscillatory solutions of three dimensional time-scale systems

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Abstract: In this article, we classify nonoscillatory solutions of a system of three-dimensional time scale systems. We use the method of considering the sign of components of such solutions. Examples are given to highlight some of our results. Moreover, the existence of such solutions is obtained by Knaster's fixed point theorem.

Key words: Time scales, oscillation, three-dimensional systems

1. Introduction

We consider the system

$$\begin{cases} x^{\Delta}(t) = a(t)f(y(t)) \\ y^{\Delta}(t) = b(t)g(z(t)) \\ z^{\Delta}(t) = -c(t)h(x(t)) \end{cases} \tag{1.1}$$

on a time scale $\mathbb{T}$, i.e. a nonempty closed subset of real numbers, where $a, b : \mathbb{T} \mapsto [0, \infty)$ (not identically zero) and $c : \mathbb{T} \mapsto (0, \infty)$ are rd-continuous functions and $f, g, h : \mathbb{R} \mapsto \mathbb{R}$ are continuous functions satisfying $uf(u) > 0$, $ug(u) > 0$, and $uh(u) > 0$ for $u \neq 0$ and

$$\frac{f(u)}{\Phi_\alpha(u)} \geq F, \quad \frac{g(u)}{\Phi_\beta(u)} \geq G, \quad \frac{h(u)}{\Phi_\gamma(u)} \geq H \quad \text{for all } u \neq 0, \tag{1.2}$$

where F, G, and H are positive constants and $\Phi_p(u) = |u|^p \operatorname{sgn} u$, $p > 0$ and $p \in \{\alpha, \beta, \gamma\}$ is an odd power function. Here, we define *rd-continuity* as that it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist at left-dense points in $\mathbb{T}$. Throughout this paper, we consider only unbounded time scales. A solution (x, y, z) defined on $[t_0, \infty) \subset \mathbb{T}$, $t_0 \in \mathbb{T}$, is called *proper* provided $\sup\big\{|x(s)|, |y(s)|, |z(s)| : s \in [t, \infty)\big\} > 0$ for $t \geq t_0$. A proper solution of system (1.1) is called oscillatory if all of its components x, y, z are oscillatory, i.e. neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory. It is so clear to see that if one component of a solution (x, y, z) is eventually of one sign, then all its components are eventually of one sign, see [3]. Therefore, nonoscillatory solutions have all components nonoscillatory.

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For convenience let us set

$$I_a = \int_T^\infty a(t)\Delta t, \quad I_b = \int_T^\infty b(t)\Delta t, \quad I_c = \int_T^\infty c(t)\Delta t, \quad T \in \mathbb{T}.$$

A special case of system (1.1)

$$\begin{cases} x^\Delta(t) = a(t)y^\alpha(t) \\ y^\Delta(t) = b(t)z^\beta(t) \\ z^\Delta(t) = -c(t)x^\gamma(t) \end{cases}$$

is considered by Akın et al. in [2] when $I_a = I_b = \infty$ holds and the oscillatory properties of the system are investigated. After that, Akın et al. also consider the system

$$\begin{cases} x^\Delta(t) = a(t)f(y(t)) \\ y^\Delta(t) = b(t)g(z(t)) \\ z^\Delta(t) = \pm c(t)h(x(t)) \end{cases}$$

in [3] and classify the nonoscillatory solutions of the system above under the conditions (1.2) and $I_a = I_b = \infty$. Specifically, [3, Theorem 4.3] shows us that every nonoscillatory solution of system (1.1) is a Kneser solution when $\alpha\beta\gamma < 1$. The case $\alpha\beta\gamma \geq 1$ is left as an open problem in [3]. In this paper, we do not only solve the open problem but also we remove the strict condition $I_a = I_b = \infty$. Consequently, we have to deal with four types of nonoscillatory solutions instead of two. In addition to that, we obtain the existence of nonoscillatory solutions of system (1.1) which is not studied in [3]. Some other versions of two and three dimensional time scale systems and delay time–scale systems are considered in [1, 11–13], respectively. We also suggest [4] for the continuous case, [7, 8, 10, 14] for the discrete case and the books [5, 6] by Bohner and Peterson about the theory of time scales.

If (1.1) has a nonoscillatory solution (x, y, z), then there are four types of such a solution of (1.1), namely

$$\begin{aligned} \text{Type (a):} \quad & \operatorname{sgn} x(t) = \operatorname{sgn} y(t) = \operatorname{sgn} z(t), \\ \text{Type (b):} \quad & \operatorname{sgn} x(t) = \operatorname{sgn} z(t) \neq \operatorname{sgn} y(t), \\ \text{Type (c):} \quad & \operatorname{sgn} x(t) = \operatorname{sgn} y(t) \neq \operatorname{sgn} z(t), \\ \text{Type (d):} \quad & \operatorname{sgn} x(t) \neq \operatorname{sgn} y(t) = \operatorname{sgn} z(t). \end{aligned}$$

We eliminate nonoscillatory solutions of Types (a), (c), and (d) by integral conditions of a, b, and c. Elimination is shown by single and double integrals in Section 2 and by triple integrals in Section 3. Section 3 is divided into three subsections depending on $\alpha\beta\gamma$. The last section is about the existence of nonoscillatory solutions and we also include two examples related with Type (b) solutions. In our proofs, we always assume that x is eventually positive.

2. Elimination by single and double integrals

In this section, we obtain single and double integrals of the coefficient functions to eliminate nonoscillatory solutions of Types (a), (c), and (d).

Theorem 2.1 *Any nonoscillatory solution of system* (1.1) *cannot be of:*

(i) *Type (a) if* $I_c = \infty$*;*

(ii) *Type (c) if* $I_b = \infty$*;*

(iii) *Type (d) if* $I_a = \infty$*.*

Proof The proof of (i) can be found in the proof of Theorem 4.1 in [3]. Hence, we only prove parts (ii) and (iii). To prove (ii), suppose that (x, y, z) is a nonoscillatory solution of system (1.1) such that $x(t), y(t)$ are positive and $z(t)$ is negative for $t \geq T$, $T \in \mathbb{T}$. The positivity of x and the third equation of system (1.1) give us that $z(t)$ is nonincreasing for $t \geq T$. Hence, there exist $T_1 \geq T$, $T_1 \in \mathbb{T}$ and $l < 0$ such that $g(z(t)) \leq l$ for $t \geq T_1$. The integration of the second equation from T_1 to t yield us

$$y(t) - y(T_1) = \int_{T_1}^{t} b(s)g(z(s))\Delta s \leq l \int_{T_1}^{t} b(s)\Delta s, \quad t \geq T_1.$$

Thus, we have that $y(t)$ diverges to negative infinity as t tends to infinity, but then this contradicts with that $y(t)$ is positive for large t. Hence, this leads us to that (x, y, z) cannot be of Type (c).

To prove (iii), we now suppose that (x, y, z) is a nonoscillatory solution of system (1.1) such that $x(t)$ is positive, $y(t)$ and $z(t)$ are negative for $t \geq T$, $T \in \mathbb{T}$. The fact that z is eventually negative and the second equation of system (1.1) give us that $y(t)$ is nonincreasing for $t \geq T$. Hence, there exist $T_1 \geq T$, $T_1 \in \mathbb{T}$, and $l < 0$ such that $f(y(t)) \leq l$ for $t \geq T_1$.

Integrating the first equation of system (1.1) from T_1 to t and using the above inequality give

$$x(t) - x(T_1) = \int_{T_1}^{t} a(s)f(y(s))\Delta s \leq l \int_{T_1}^{t} a(s)\Delta s, \quad t \geq T_1.$$

Then $x(t)$ diverges to negative infinity as t tends to infinity, but then this contradicts with $x(t) > 0$ for large t. Hence, this implies that (x, y, z) cannot be of Type (d). □

The proof of (ii) of the following theorem can be found in the proof of Theorem 4.2 in [3].

Theorem 2.2 *Any nonoscillatory solution of system* (1.1) *cannot be of*

(i) *Type (a) if*

$$\int_{T_2}^{\infty} c(s) \left(\int_{T_1}^{s} a(\tau)\Delta\tau \right)^{\gamma} \Delta s = \infty; \tag{2.1}$$

(ii) *Type (c) if*

$$\int_{T_2}^{\infty} b(s) \left(\int_{T_1}^{s} c(\tau)\Delta\tau \right)^{\beta} \Delta s = \infty; \tag{2.2}$$

(iii) *Type (d) if*

$$\int_{T_2}^{\infty} a(s) \left(\int_{T_1}^{s} b(\tau)\Delta\tau \right)^{\alpha} \Delta s = \infty. \tag{2.3}$$

Proof Suppose that (x, y, z) is such a solution of system (1.1) claimed as in the assumption.

(i) We now show that (x, y, z) cannot be of Type (a). Assume that it is, then $y(t) > 0$, $z(t) > 0$ for $t \geq T$. The positivity of z and the second equation of system (1.1) give us that $y(t)$ is nondecreasing for $t \geq T$. Thus, there exist $T_1 \geq T$, $T_1 \in \mathbb{T}$ and $k > 0$ such that

$$f(y(t)) \geq k \tag{2.4}$$

for $t \geq T_1$. The integration of the first equation of system (1.1) from T_1 to t and substitution (2.4) into the resulting equation give us

$$x(t) \geq x(t) - x(T_1) = \int_{T_1}^{t} a(s) f(y(s)) \Delta s \geq k \int_{T_1}^{t} a(s) \Delta s$$

or

$$\Phi_\gamma(x(t)) \geq k^\gamma \left(\int_{T_1}^{t} a(s) \Delta s \right)^\gamma, \quad t \geq T_1.$$

Then by (1.2) there exist $T_2 \in \mathbb{T}$, $T_2 \geq T_1$ and $H > 0$ such that

$$h(x(t)) \geq H \Phi_\gamma(x(t)) = H x^\gamma(t) \tag{2.5}$$

for $t \geq T_2$; hence,

$$h(x(t)) \geq H k^\gamma \left(\int_{T_1}^{t} a(s) \Delta s \right)^\gamma, \quad t \geq T_2.$$

The integration of the last equation of system (1.1) from T_2 to t and inequaility (2.5) yield us

$$z(t) - z(T_2) = -\int_{T_2}^{t} c(s) h(x(s)) \Delta s \leq -H k^\gamma \int_{T_2}^{t} c(s) \left(\int_{T_1}^{s} a(\tau) \Delta \tau \right)^\gamma \Delta s$$

for $t \geq T_2$. As $t \to \infty$, $z(t) \to -\infty$ by (2.1) but this is a contradiction to $z > 0$ eventually. This implies that (x, y, z) cannot be of Type (a).

(iii) We now show that (x, y, z) cannot be of Type (d). Assume that it is, then $y(t)$ and $z(t)$ are negative for $t \geq T$. The positivity of x and the last equation of system (1.1) give us that $z(t)$ is nonincreasing for $t \geq T$. Hence, there exist $T_1 \geq T$, $T_1 \in \mathbb{T}$ and $l < 0$ such that

$$g(z(t)) \leq l \tag{2.6}$$

for $t \geq T_1$. Then, the integration of the second equation of system (1.1) from T_1 to t leads us to

$$y(t) \leq y(t) - y(T_1) = \int_{T_1}^{t} b(s) g(z(s)) \Delta s \leq l \int_{T_1}^{t} b(s) \Delta s$$

or

$$\Phi_\alpha(y(t)) \leq l^\alpha \left(\int_{T_1}^{t} b(s) \Delta s \right)^\alpha, \quad t \geq T_1.$$

Then by negativity of y and (1.2) there exist $T_2 \in \mathbb{T}$, $T_2 \geq T_1$ and $F > 0$ such that

$$f(y(t)) \leq F\ \Phi_\alpha(y(t)) \tag{2.7}$$

for $t \geq T_2$, and so

$$f(y(t)) \leq F l^\alpha \left(\int_{T_1}^t b(s)\Delta s \right)^\alpha, \quad t \geq T_2.$$

By integrating the first equation of system (1.1) from T_2 to t and taking the inequality above into account, we get

$$x(t) - x(T_2) = \int_{T_2}^t a(s) f(y(s))\Delta s \leq F l^\alpha \int_{T_2}^t a(s) \left(\int_{T_1}^s b(\tau)\Delta\tau \right)^\alpha \Delta s$$

for $t \geq T_2$. As $t \to \infty$, $x(t) \to -\infty$ by (2.3) but this contradicts with the positivity of x. This implies that (x, y, z) cannot be of Type (c). This completes the proof. □

3. Elimination by triple integrals

In this section, we obtain triple integrals of the coefficient functions to eliminate Types (a), (c), and (d) solutions of system (1.1). In order to achieve this, we divide this section into three subsections regarding whether $\alpha\beta\gamma < 1$, $\alpha\beta\gamma = 1$ and $\alpha\beta\gamma > 1$.

The following lemma plays an important role to prove our results in this section. The proof is based on the chain rule on time scales, see Theorem 1.90 in [5]. Part (a) is used in the case $\alpha\beta\gamma \leq 1$ while Part (b) is necessary in the case $\alpha\beta\gamma > 1$.

Lemma 3.1 *Let* $y \in C^1_{rd}(\mathbb{T}, \mathbb{R}^+)$.

Let $0 < \eta < 1$. *If* $y^\Delta(t) < 0$ *on* $\mathbb{T}$, *then*

$$\int_T^\infty \frac{y^\Delta(t)}{y^\eta(\sigma(t))}\Delta t < \infty, \quad T \in \mathbb{T}.$$

(b) *Let* $\eta > 1$. *If* $y^\Delta(t) > 0$ *on* $\mathbb{T}$, *then*

$$\int_T^\infty \frac{y^\Delta(t)}{y^\eta(\sigma(t))}\Delta t < \infty, \quad T \in \mathbb{T}.$$

Proof The proof of (b) can be found in [2, Remark 4.2]. Therefore, we only prove (a) here. By the chain rule on time scales, we have

$$\left(y^{1-\eta}(t)\right)^\Delta = (1-\eta) y^\Delta(t) \int_0^1 \frac{1}{(y(t) + \mu(t) h y^\Delta(t))^\eta} dh.$$

Since $0 < y^\sigma = y + \mu y^\Delta \leq y + \mu h y^\Delta \leq y$, we have

$$\frac{y^\Delta(t)}{y^\eta(\sigma(t))} \leq \frac{1}{1-\eta}\left(y^{1-\eta}(t)\right)^\Delta. \tag{3.1}$$

The integration of inequality (3.1) from T to t leads us

$$\int_T^t \frac{y^{\Delta}(s)}{y^{\eta}(\sigma(s))}\Delta s \le \frac{1}{1-\eta}\left[y^{1-\eta}(t) - y^{1-\eta}(T)\right].$$

Since $1-\eta > 0$, and y is decreasing, we obtain that

$$\int_T^{\infty} \frac{y^{\Delta}(s)}{y^{\eta}(\sigma(s))}\Delta s < \infty.$$

□

3.1. The case $\alpha\beta\gamma < 1$

Theorems in this section are shown by Lemma 3.1 (a) and the proof of (i) can be found in Theorem 4.3 in [3].

Theorem 3.2 *Any nonoscillatory solution of system* (1.1) *cannot be of*

(i) *Type (a) if*

$$\int_{T_3}^{\infty} c(s)\left(\int_{T_2}^{s} a(\tau)\left(\int_{T_1}^{\tau} b(\upsilon)\Delta\upsilon\right)^{\alpha}\Delta\tau\right)^{\gamma}\Delta s = \infty; \tag{3.2}$$

(ii) *Type (c) if*

$$\int_{T_3}^{\infty} b(s)\left(\int_{T_2}^{s} c(\tau)\left(\int_{T_1}^{\tau} a(\upsilon)\Delta\upsilon\right)^{\gamma}\Delta\tau\right)^{\beta}\Delta s = \infty; \tag{3.3}$$

(iii) *Type (d) if*

$$\int_{T_3}^{\infty} a(s)\left(\int_{T_2}^{s} b(\tau)\left(\int_{T_1}^{\tau} c(\upsilon)\Delta\upsilon\right)^{\beta}\Delta\tau\right)^{\alpha}\Delta s = \infty. \tag{3.4}$$

Proof Suppose that (x, y, z) is such a solution of system (1.1), claimed as in the theorem.

(ii) We now show that (x, y, z) cannot be of Type (c). Assume that it is, then $y(t) > 0$, $z(t) < 0$ for $t \ge T$. Then by (1.2) there exist $T_1 \in \mathbb{T}$, $T_1 \ge T$ and $F > 0$ such that

$$f(y(t)) \ge F\; y^{\alpha}(t) \tag{3.5}$$

for $t \ge T_1$. Integration of the first equation of system (1.1) from T_1 to t and taking (3.5) into account yield for $t \ge T_1$,

$$x(t) \ge x(t) - x(T_1) \ge F\int_{T_1}^{t} a(s)y^{\alpha}(s)\Delta s \ge F\; y^{\alpha}(t)\int_{T_1}^{t} a(s)\Delta s,$$

where we use the monotonicity of y. This leads us to

$$x^{\gamma}(t) \ge F^{\gamma}\; y^{\alpha\gamma}(t)\left(\int_{T_1}^{t} a(s)\Delta s\right)^{\gamma}, \quad t \ge T_1. \tag{3.6}$$

Then we can find $T_2 \in \mathbb{T}$, $T_2 \geq T_1$ and $H > 0$ such that (2.5) holds. Substituting (3.6) into (2.5) yields

$$h(x(t)) \geq H\ F^{\gamma}\ y^{\alpha\gamma}(t) \left(\int_{T_1}^{t} a(s)\Delta s \right)^{\gamma}, \quad t \geq T_2.$$

Integration of the last equation of system (1.1) from T_2 to t and the above inequality lead us to

$$z(t) \leq z(t) - z(T_2) \leq -H\ F^{\gamma}\ y^{\alpha\gamma}(t) \int_{T_2}^{t} c(s) \left(\int_{T_1}^{s} a(\tau)\Delta\tau \right)^{\gamma} \Delta s$$

or

$$\Phi_{\beta}\left(z(t)\right) \leq -H^{\beta}\ F^{\beta\gamma}\ y^{\alpha\beta\gamma}(t) \left(\int_{T_2}^{t} c(s) \left(\int_{T_1}^{s} a(\tau)\Delta\tau \right)^{\gamma} \Delta s \right)^{\beta},\ t \geq T_2. \tag{3.7}$$

Then by (1.2) there exist $T_3 \in \mathbb{T}$, $T_3 \geq T_2$ and $G > 0$ such that

$$g(z(t)) \leq G\Phi_{\beta}(z(t)) \tag{3.8}$$

holds for $t \geq T_3$. Substituting (3.7) into (3.8) we have

$$g(z(t)) \leq -G\ H^{\beta}\ F^{\beta\gamma}\ y^{\alpha\beta\gamma}(t) \left(\int_{T_2}^{t} c(s) \left(\int_{T_1}^{s} a(\tau)\Delta\tau \right)^{\gamma} \Delta s \right)^{\beta}, \tag{3.9}$$

$t \geq T_3$. Also the second equation of (1.1) and (3.9) give us

$$y^{\Delta}(t) \leq -G\ H^{\beta}\ F^{\beta\gamma}\ y^{\alpha\beta\gamma}(\sigma(t))b(t) \left(\int_{T_2}^{t} c(s) \left(\int_{T_1}^{s} a(\tau)\Delta\tau \right)^{\gamma} \Delta s \right)^{\beta},$$

where we use the monotonicity of y. Dividing both sides of the inequality above by $x^{\alpha\beta\gamma}(\sigma(t))$ and the integration from T_3 to t yield

$$\int_{T_3}^{t} \frac{y^{\Delta}(s)}{y^{\alpha\beta\gamma}(\sigma(s))}\Delta s \leq -G\ H^{\beta}\ F^{\beta\gamma}\ \times \int_{T_3}^{t} b(s) \left(\int_{T_2}^{s} c(\tau) \left(\int_{T_1}^{\tau} a(\upsilon)\Delta\upsilon \right)^{\gamma} \Delta\tau \right)^{\beta} \Delta s, \quad t \geq T_3.$$

As $t \to \infty$,

$$\int_{T_3}^{\infty} \frac{y^{\Delta}(s)}{y^{\alpha\beta\gamma}(\sigma(s))}\Delta s = -\infty$$

by (3.3). On the other hand, Lemma 3.1 (a) gives us a contradiction.

(iii) We now show that (x, y, z) cannot be of Type (d). Assume that it is, then $y(t) < 0$, $z(t) < 0$ for $t \geq T$. Then by (1.2) there exist $T_1 \in \mathbb{T}$, $T_1 \geq T$ and $H > 0$ such that (2.5) holds. By the integration of the last equation of system (1.1) from T_1 to t and by (2.5), we have for $t \geq T_1$,

$$z(t) \leq z(t) - z(T_1) \leq -H \int_{T_1}^{t} c(s)x^{\gamma}(s)\Delta s \leq -H\ x^{\gamma}(t) \int_{T_1}^{t} c(s)\Delta s,$$

where we use the fact that x is monotonic. That leads us to

$$\Phi_{\beta}\left(z(t)\right) \leq -H^{\beta}\ x^{\beta\gamma}(t) \left(\int_{T_1}^{t} c(s)\Delta s \right)^{\beta}, \quad t \geq T_1. \tag{3.10}$$

Then there exist $T_2 \in \mathbb{T}$, $T_2 \geq T_1$ and $G > 0$ such that

$$g(z(t)) \leq G\ \Phi_\beta\,(z(t)) \tag{3.11}$$

for $t \geq T_2$. Substituting (3.10) into (3.11) yields

$$g(z(t)) \leq -G\ H^\beta\ x^{\beta\gamma}(t)\left(\int_{T_1}^{t} c(s)\Delta s\right)^\beta, \quad t \geq T_2.$$

Integrating the middle equation of system (1.1) from T_2 to t and taking the above inequality into account give us

$$y(t) \leq y(t) - y(T_2) \leq -G\ H^\beta\ x^{\beta\gamma}(t)\int_{T_2}^{t} b(s)\left(\int_{T_1}^{s} c(\tau)\Delta\tau\right)^\beta \Delta s$$

or

$$\Phi_\alpha\,(y(t)) \leq -G^\alpha\ H^{\alpha\beta}\ x^{\alpha\beta\gamma}(t)\left(\int_{T_2}^{t} b(s)\left(\int_{T_1}^{s} c(\tau)\Delta\tau\right)^\beta \Delta s\right)^\alpha,\ t \geq T_2.$$

Then by (1.2), there exist $T_3 \in \mathbb{T}$, $T_3 \geq T_2$ and $F > 0$ such that for $t \geq T_3$,

$$\begin{aligned} f(y(t)) &\leq F\ \Phi_\alpha\,(y(t)) \\ &\leq -FG^\alpha\ H^{\alpha\beta}\ x^{\alpha\beta\gamma}(t)\left(\int_{T_2}^{t} b(s)\left(\int_{T_1}^{s} c(\tau)\Delta\tau\right)^\beta \Delta s\right)^\alpha. \end{aligned} \tag{3.12}$$

By using the first equation of system (1.1) and (3.12), one gets

$$x^\Delta(t) \leq -FG^\alpha\ H^{\alpha\beta}\ x^{\alpha\beta\gamma}\,(\sigma(t))\,a(t)\left(\int_{T_2}^{t} b(s)\left(\int_{T_1}^{s} c(\tau)\Delta\tau\right)^\beta \Delta s\right)^\alpha,$$

where we use the monotonicity of x. By a simple algebra on the inequality above and the integration of that inequality from T_3 to t yield

$$\int_{T_3}^{t} \frac{x^\Delta(s)}{x^{\alpha\beta\gamma}\,(\sigma(s))}\Delta s \leq -FG^\alpha\ H^{\alpha\beta}\int_{T_3}^{t} a(s)\left(\int_{T_2}^{s} b(\tau)\left(\int_{T_1}^{\tau} c(\upsilon)\Delta\upsilon\right)^\beta \Delta\tau\right)^\alpha \Delta s, \quad t \geq T_3.$$

As $t \to \infty$,

$$\int_{T_3}^{\infty} \frac{x^\Delta(s)}{x^{\alpha\beta\gamma}\,(\sigma(s))}\Delta s = -\infty$$

by (3.4). On the other hand, Lemma 3.1 (a) gives us a contradiction. Therefore, the proof is completed. $\square$

3.2. The case $\alpha\beta\gamma = 1$

Theorems in this section are shown by Lemma 3.1 (a) as well since we use the proofs of Theorem 3.2. Therefore, we will skip some of the proofs in this section.

Theorem 3.3 *Let $0 < \epsilon < 1$. Any nonoscillatory solution of system (1.1) cannot be of*

(i) *Type (a) if*

$$\int_{T_3}^{\infty} c(s) \left(\int_{T_2}^{s} a(\tau) \left(\int_{T_1}^{\tau} b(\upsilon)\Delta\upsilon\right)^{\alpha} \Delta\tau\right)^{\gamma(1-\epsilon)} \Delta s = \infty; \tag{3.13}$$

(ii) *Type (c) if*

$$\int_{T_3}^{\infty} b(s) \left(\int_{T_2}^{s} c(\tau) \left(\int_{T_1}^{\tau} a(\upsilon)\Delta\upsilon\right)^{\gamma} \Delta\tau\right)^{\beta(1-\epsilon)} \Delta s = \infty, \tag{3.14}$$

(iii) *Type (d) if*

$$\int_{T_3}^{\infty} a(s) \left(\int_{T_2}^{s} b(\tau) \left(\int_{T_1}^{\tau} c(\upsilon)\Delta\upsilon\right)^{\beta} \Delta\tau\right)^{\alpha(1-\epsilon)} \Delta s = \infty. \tag{3.15}$$

Proof Let $0 < \epsilon < 1$ and suppose that (x, y, z) is a nonoscillatory solution of system (1.1).

(i) We now show that (x, y, z) cannot be of Type (a). Assume that it is, then $y(t) > 0$, $z(t) > 0$ for $t \geq T$. As shown in the proof of Theorem 4.3 in [3], there exist $T_2 \geq T_1 \geq T$ and $F, G > 0$ such that

$$x(t) \geq F\ G^{\alpha}\ z^{\alpha\beta}(\sigma(t)) \int_{T_2}^{t} a(s) \left(\int_{T_1}^{s} b(\tau)\Delta\tau\right)^{\alpha} \Delta s,$$

where we use the monotonicity of z and so

$$(x(t))^{\gamma(1-\epsilon)} \geq F^{\gamma(1-\epsilon)}\ G^{\alpha\gamma(1-\epsilon)}\ z^{(1-\epsilon)}(\sigma(t)) \left(\int_{T_2}^{t} a(s) \left(\int_{T_1}^{s} b(\tau)\Delta\tau\right)^{\alpha} \Delta s\right)^{\gamma(1-\epsilon)}.$$

From the fact that x is positive and monotonic, one can get

$$(x(t))^{\gamma(1-\epsilon)} \leq \frac{1}{k^{\epsilon}}(x(t))^{\gamma}, \quad k > 0.$$

Hence,

$$x^{\gamma}(t) \geq k^{\epsilon} F^{\gamma(1-\epsilon)}\ G^{\alpha\gamma(1-\epsilon)}\ z^{(1-\epsilon)}(\sigma(t)) \left(\int_{T_2}^{t} a(s) \left(\int_{T_1}^{s} b(\tau)\Delta\tau\right)^{\alpha} \Delta s\right)^{\gamma(1-\epsilon)}.$$

Then by (1.2), there exist $T_3 \in \mathbb{T}$, $T_3 \geq T_2$ and $H > 0$ such that (2.5) holds for $t \geq T_3$. This implies that

$$h(x(t)) \geq H k^{\epsilon} F^{\gamma(1-\epsilon)}\ G^{\alpha\gamma(1-\epsilon)}\ z^{(1-\epsilon)}(\sigma(t)) \left(\int_{T_2}^{t} a(s) \left(\int_{T_1}^{s} b(\tau)\Delta\tau\right)^{\alpha} \Delta s\right)^{\gamma(1-\epsilon)} \tag{3.16}$$

for $t \geq T_3$. Taking the last equation of system (1.1) and (3.16) into account yield us

$$z^{\Delta}(t) \leq -H k^{\epsilon} F^{\gamma(1-\epsilon)}\ G^{\alpha\gamma(1-\epsilon)}\ z^{(1-\epsilon)}(\sigma(t))\, c(t) \left(\int_{T_2}^{t} a(s) \left(\int_{T_1}^{s} b(\tau)\Delta\tau\right)^{\alpha} \Delta s\right)^{\gamma(1-\epsilon)}.$$

Integration of the above inequality from T_3 to t after dividing it by $z^{(1-\epsilon)}(\sigma(t))$ yield

$$\int_{T_3}^{t} \frac{z^{\Delta}(s)}{z^{(1-\epsilon)}(\sigma(s))}\Delta s \leq -H k^{\epsilon} F^{\gamma(1-\epsilon)}\ G^{\alpha\gamma(1-\epsilon)} \int_{T_3}^{t} c(s) \left(\int_{T_2}^{s} a(\tau) \left(\int_{T_1}^{\tau} b(\upsilon)\Delta\upsilon\right)^{\alpha} \Delta\tau\right)^{\gamma(1-\epsilon)} \Delta s, \quad t \geq T_3.$$

As $t \to \infty$,

$$\int_{T_3}^{\infty} \frac{z^{\Delta}(s)}{z^{(1-\epsilon)}(\sigma(s))}\Delta s = -\infty$$

by (3.13). On the other hand, Lemma 3.1 (a) gives us a contradiction.

(ii) We now show that (x, y, z) cannot be of Type (c). Assume it is, then $y(t)$ is positive, $z(t)$ is negative for $t \geq T$. As shown in the proof of Theorem 3.2 (ii), there exist $T_2 \geq T_1 \geq T$ and $H, F > 0$ such that

$$z(t) \leq -H\ F^{\gamma}\ y^{\alpha\gamma}\left(\sigma(t)\right) \int_{T_2}^{t} c(s) \left(\int_{T_1}^{s} a(\tau)\Delta\tau\right)^{\gamma} \Delta s;$$

thus,

$$\Phi_{\beta(1-\epsilon)}\left(z(t)\right) \leq -H^{\beta(1-\epsilon)}F^{\beta\gamma(1-\epsilon)}y^{(1-\epsilon)}(\sigma(t)) \left(\int_{T_2}^{t} c(s) \left(\int_{T_1}^{s} a(\tau)\Delta\tau\right)^{\gamma} \Delta s\right)^{\beta(1-\epsilon)}.$$

From the fact that z is negative and monotonic, one gets

$$\Phi_{\beta(1-\epsilon)}\left(z(t)\right) \geq \frac{1}{k^{\epsilon}}\Phi_{\beta}\left(z(t)\right), \quad k > 0.$$

Thus,

$$\Phi_{\beta}\left(z(t)\right) \leq -k^{\epsilon}H^{\beta(1-\epsilon)}F^{\beta\gamma(1-\epsilon)}y^{1-\epsilon}(\sigma(t)) \left(\int_{T_2}^{t} c(s) \left(\int_{T_1}^{s} a(\tau)\Delta\tau\right)^{\gamma} \Delta s\right)^{\beta(1-\epsilon)}. \tag{3.17}$$

Therefore, by the second equation of system (1.1), (1.2), and (3.17) there exist $T_3 \in \mathbb{T}$, $T_3 \geq T_2$, and $G > 0$ such that

$$y^{\Delta}(t) \leq -Gk^{\epsilon}H^{\beta(1-\epsilon)}F^{\beta\gamma(1-\epsilon)}y^{1-\epsilon}(\sigma(t))b(t) \left(\int_{T_2}^{t} c(s) \left(\int_{T_1}^{s} a(\tau)\Delta\tau\right)^{\gamma} \Delta s\right)^{\beta(1-\epsilon)}.$$

The integration of the inequality above from T_3 to t after dividing it by $y^{1-\epsilon}(\sigma(t))$ yield

$$\int_{T_3}^{t} \frac{y^{\Delta}(s)}{y^{1-\epsilon}(\sigma(s))}\Delta s \leq -Gk^{\epsilon}H^{\beta(1-\epsilon)}F^{\beta\gamma(1-\epsilon)} \int_{T_3}^{t} b(s) \left(\int_{T_2}^{s} c(\tau) \left(\int_{T_1}^{\tau} a(\upsilon)\Delta\upsilon\right)^{\gamma} \Delta\tau\right)^{\beta(1-\epsilon)} \Delta s,\ t \geq T_3.$$

As $t \to \infty$, $\int_{T_3}^{\infty} \frac{y^{\Delta}(s)}{y^{1-\epsilon}(\sigma(s))}\Delta s = -\infty$ by (3.14). However, $\int_{T}^{\infty} \frac{y^{\Delta}(s)}{y^{1-\epsilon}(\sigma(s))}\Delta s < \infty$ by Lemma 3.1 (i). However, this implies that we have a contradiction and this proves assertion (ii).

(iii) can be shown similarly by using Lemma 3.1 (a) and hence we omit it. □

The following corollary is easily obtained by using the proof of Theorem 3.2. Here, we only prove the following.

Corollary 3.4 *Any nonoscillatory solution of system (1.1) cannot be of Type (a) if one of the following holds:*

$$\limsup_{t\to\infty} HF^{\gamma}G^{\alpha\gamma} \int_{t}^{\infty} c(\tau)\Delta\tau \left(\int_{T_2}^{t} a(s) \left(\int_{T_1}^{s} b(\upsilon)\Delta\upsilon\right)^{\alpha} \Delta s\right)^{\gamma} > 1; \tag{3.18}$$

$$\limsup_{t\to\infty} FG^{\alpha}H^{\alpha\beta} \left(\int_{t}^{\infty} c(\tau)\Delta\tau\right)^{\alpha\beta} \left(\int_{T_2}^{t} a(s) \left(\int_{T_1}^{s} b(\upsilon)\Delta\upsilon\right)^{\alpha} \Delta s\right)^{\gamma} > 1. \tag{3.19}$$

Proof Assume that (x,y,z) is a Type (a) solution of system (1.1) such that $x(t), y(t)$, and $z(t)$ are positive for $t \geq T$. Then (2.5) holds for $t \geq T_2$. From the integration of the last equation of system (1.1) from t to ∞ and (2.5), one gets

$$z(t) \geq \int_t^\infty c(\tau)h(x(\tau))\Delta\tau \geq H\int_t^\infty c(\tau)x^\gamma(\tau)\Delta\tau \geq Hx^\gamma(t)\int_t^\infty c(\tau)\Delta\tau. \tag{3.20}$$

At the same time there exists $T_3 \in \mathbb{T}$, $T_3 \geq T_2$ such that the following holds:

$$x(t) \geq FG^\alpha(z(t))^{\alpha\beta}\int_{T_2}^t a(\tau)\left(\int_{T_1}^\tau b(s)\Delta s\right)^\alpha \Delta\tau, \quad t \geq t \geq T_3, \tag{3.21}$$

see Theorem 4.3 in [3]. Substituting (3.21) into (3.20) we have

$$z(t) \geq HF^\gamma G^{\alpha\gamma}z(t)\left(\int_{T_2}^t a(s)\left(\int_{T_1}^s b(\tau)\Delta\tau\right)^\alpha \Delta s\right)^\gamma \int_t^\infty c(\tau)\Delta\tau, \quad t \geq T_3.$$

This implies that

$$1 \geq HF^\gamma G^{\alpha\gamma}\left(\int_{T_2}^t a(s)\left(\int_{T_1}^s b(\tau)\Delta\tau\right)^\alpha \Delta s\right)^\gamma \int_t^\infty c(\tau)\Delta\tau, \quad t \geq T_3$$

or

$$\limsup_{t\to\infty} HF^\gamma G^{\alpha\gamma}\left(\int_{T_2}^t a(s)\left(\int_{T_1}^s b(\tau)\Delta\tau\right)^\alpha \Delta s\right)^\gamma \int_t^\infty c(\tau)\Delta\tau \leq 1,$$

but this contradicts with (3.18). Substituting (3.20) into (3.21) contradicts with (3.19). So this completes the proof. □

One can find the necessary conditions for Types (c) and (d) similarly.

3.3. The case $\alpha\beta\gamma > 1$

Lemma 3.1 (b) is needed to prove our main results we obtain in this section.

Theorem 3.5 *Any nonoscillatory solution of system (1.1) cannot be of*

(i) *Type (a) if*

$$\int_{T_2}^\infty a(s)\left(\int_{\sigma(s)}^\infty c(\tau)\Delta\tau\right)^{\beta\alpha}\left(\int_{T_1}^s b(\upsilon)\Delta\upsilon\right)^\alpha \Delta s = \infty; \tag{3.22}$$

(ii) *Type (c) if*

$$\int_{T_2}^\infty c(s)\left(\int_{\sigma(s)}^\infty b(\tau)\Delta\tau\right)^{\alpha\gamma}\left(\int_{T_1}^s a(\upsilon)\Delta\upsilon\right)^\gamma \Delta s = \infty; \tag{3.23}$$

(iii) *Type (d) if*

$$\int_{T_2}^\infty b(s)\left(\int_{\sigma(s)}^\infty a(\tau)\Delta\tau\right)^{\beta\gamma}\left(\int_{T_1}^s c(\upsilon)\Delta\upsilon\right)^\beta \Delta s = \infty. \tag{3.24}$$

Proof Let (x, y, z) be such a solution of system (1.1) claimed as in the theorem.

(i) We now show that (x, y, z) cannot be of Type (a). Assume that it is, then $y(t) > 0$, $z(t) > 0$ for $t \geq T$. Then there exist $T_1 \geq T$, $T_1 \in \mathbb{T}$ and $H > 0$ such that (2.5) holds. The integration of the last equation of system (1.1) from $\sigma(t)$ to ∞, using (2.5) and the fact that x is nondecreasing show us that

$$\begin{aligned} z(\sigma(t)) &\geq \int_{\sigma(t)}^{\infty} c(s)h(x(s))\Delta s \\ &\geq H\ (x(\sigma(t)))^{\gamma} \int_{\sigma(t)}^{\infty} c(s)\Delta s \end{aligned}$$

or

$$z^{\beta}(\sigma(t)) \geq H^{\beta}\ (x(\sigma(t)))^{\beta\gamma} \left(\int_{\sigma(t)}^{\infty} c(s)\Delta s\right)^{\beta},\ t \geq T_1. \tag{3.25}$$

Also, there exist $G > 0$ and $T_2 \geq T_1$, $T_2 \in \mathbb{T}$ such that

$$g(z(t)) \geq G\ \Phi_{\beta}(z(t)) \tag{3.26}$$

for $t \geq T_2$. Integration of the middle equation of system (1.1) from T_2 to t and using the facts that z is nonincreasing, positive, y is positive eventually and (3.26) lead us to

$$y(t) \geq \int_{T_2}^{t} b(s)g\left(z\left(s\right)\right)\Delta s \geq G\ z^{\beta}(\sigma(t)) \int_{T_2}^{t} b(s)\Delta s, \quad t \geq T_2. \tag{3.27}$$

By substituting (3.25) into (3.27) and taking α^{th} power of the resulting inequality, we have

$$y^{\alpha}(t) \geq G^{\alpha} H^{\alpha\beta}\ (x(\sigma(t)))^{\alpha\beta\gamma} \left(\int_{\sigma(t)}^{\infty} c(s)\Delta s\right)^{\alpha\beta} \left(\int_{T_2}^{t} b(s)\Delta s\right)^{\alpha} \tag{3.28}$$

for $t \geq T_2$. From the positivity of y, there exist $F > 0$ and $T_3 \geq T_2$, $T_2 \in \mathbb{T}$ such that (3.5) holds. Using this fact and substituting (3.28) into the first equation of system (1.1) we have for $t \geq T_3$,

$$x^{\Delta}(t) \geq FG^{\alpha} H^{\alpha\beta}\ (x(\sigma(t)))^{\alpha\beta\gamma} a\left(t\right) \left(\int_{\sigma(t)}^{\infty} c(s)\Delta s\right)^{\alpha\beta} \left(\int_{T_2}^{t} b(s)\Delta s\right)^{\alpha}.$$

Dividing the above inequality by $(x(\sigma(t)))^{\alpha\beta\gamma}$ and integrating the resulting inequality from T_3 to t yield

$$\int_{T_3}^{t} \frac{x^{\Delta}(s)}{(x(\sigma(s)))^{\alpha\beta\gamma}}\Delta s \geq FG^{\alpha} H^{\alpha\beta} \int_{T_3}^{t} a\left(s\right) \left(\int_{\sigma(s)}^{\infty} c(\tau)\Delta \tau\right)^{\alpha\beta} \left(\int_{T_2}^{s} b(v)\Delta v\right)^{\alpha} \Delta s.$$

By (3.22), we have $\displaystyle\int_{T_3}^{\infty} \frac{x^{\Delta}(s)}{(x(\sigma(s)))^{\alpha\beta\gamma}}\Delta s = \infty$, but this contradicts with Lemma 3.1 (b), and finishes the proof of (i).

In order to show (ii), we first integrate the middle equation of system (1.1) from $\sigma(t)$ to infinity, and then use the third and second equations of system (1.1) to arrive at

$$\int_{T_3}^{t} \frac{(-z(s))^{\Delta}}{(-z(\sigma(s)))^{\alpha\beta\gamma}} \Delta s \geq HF^{\gamma}G^{\alpha\gamma} \int_{T_3}^{t} c(s) \left(\int_{\sigma(s)}^{\infty} b(\tau)\Delta\tau \right)^{\alpha\gamma} \left(\int_{T_2}^{s} a(\upsilon)\Delta\upsilon \right)^{\gamma} \Delta s,$$

for $t \geq T_3 \geq T_2$, $T_2, T_3 \in \mathbb{T}$. By (3.23), we have $\displaystyle\int_{T_3}^{\infty} \frac{(-z(s))^{\Delta}}{(-z(\sigma(s)))^{\alpha\beta\gamma}} \Delta s = \infty$, but this contradicts with Lemma 3.1 (b), and completes the proof of (ii).

In order to show (iii), we integrate the first equation from $\sigma(t)$ to ∞ and use the last equation and then the middle equation of system (1.1) to arrive at

$$\int_{T_3}^{t} \frac{(-y(s))^{\Delta}}{(-y(\sigma(s)))^{\beta\alpha\gamma}} \Delta s \geq GH^{\beta}F^{\beta\gamma} \int_{T_3}^{t} b(s) \left(\int_{\sigma(s)}^{\infty} a(\tau)\Delta\tau \right)^{\beta\gamma} \left(\int_{T_2}^{s} c(\upsilon)\Delta\upsilon \right)^{\beta} \Delta s$$

for $t \geq T_3 \geq T_2$, $T_2, T_3 \in \mathbb{T}$. By (3.24), we have $\displaystyle\int_{T_3}^{\infty} \frac{(-y(s))^{\Delta}}{(-y(\sigma(s)))^{\beta\alpha\gamma}} \Delta s = \infty$, but this contradicts with Lemma 3.1 (b), and so the proof of (iii) is completed. □

4. Conclusions

If we choose suitable conditions from Theorems 2.1–3.5, then we can conclude that every solution of system (1.1) is either oscillatory or of Type (b). For instance, if $I_a = I_b = I_c = \infty$ hold or (2.1), (2.2), and (2.3) hold or $I_c = \infty$, (2.2), and (2.3) hold or $I_a = \infty$, (2.1), (2.2) hold or $I_b = I_a = \infty$ and (3.2) with $\alpha\beta\gamma < 1$ hold, then any nonoscillatory solution of system (1.1) belongs to Type (b).

We now consider the following continuous and discrete systems given us exact Type (b) solutions of the systems.

Example 4.1 *Let us examine*

$$\begin{cases} x^{\Delta} = e^{t}y \\ y^{\Delta} = 2e^{4t}z \\ z^{\Delta} = -6e^{-3t}x^{3}. \end{cases} \tag{4.1}$$

with $\mathbb{T} = \mathbb{R}$. *Since* $I_a = I_b = \infty$, *Theorem 2.1 (ii) and (iii) indicate that solutions of Types (c) and (d) are eliminated. Moreover, since* $I_c < \infty$ *and* (3.22) *holds in Theorem 3.5, there will no solutions of Type (a) either. Therefore, every nonoscillatory solution of system* (4.1) *belongs to Type (b) and* $(e^{-t}, -e^{-2t}, e^{-6t})$ *is such a solution.*

Example 4.2 *Now, let us view*

$$\begin{cases} x^{\Delta} = \dfrac{q^{1/6}-1}{(q-1)(1+\sqrt{q})q^{5/3}t^{2/3}} |y| \operatorname{sgn} y \\ y^{\Delta} = \frac{1}{\sqrt{t}} |z| \operatorname{sgn} z \\ z^{\Delta} = \frac{1}{t^{3/2}} |x|^{3} \operatorname{sgn} x \end{cases} \tag{4.2}$$

with $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$ *and show* $I_a = I_b = \infty$, *and* $I_c < \infty$. *Indeed,*

$$\int_1^t a(s)\Delta s = \sum_{s\in[1,t)_{q^{\mathbb{N}_0}}} \frac{q^{1/6}-1}{(q-1)(1+\sqrt{q})q^{5/3}s^{2/3}} \cdot (q-1)s = k_1 \sum_{s\in[1,t)_{q^{\mathbb{N}_0}}} s^{\frac{1}{3}},$$

where $k_1 = \frac{q^{1/6}-1}{(1+\sqrt{q})q^{5/3}}$. *By using the fact* $s = q^n$, *we have*

$$I_a = \lim_{m\to\infty} \sum_{n=0}^{m} q^{\frac{n}{3}} = \infty.$$

Similarly, it can be shown that $I_b = \infty$ *and* $I_c < \infty$. *Therefore, by Theorem 2.1 Types (d) and (c) solutions are eliminated. Next, we show that Equation (3.22) holds so that the nonoscillatory solution cannot also be of Type (a). First note that*

$$\int_1^s b(\nu)\Delta\nu = \sum_{\nu\in[1,s)_{q^{\mathbb{N}_0}}} \frac{1}{\sqrt{v}} \cdot (q-1)\nu \geq \sqrt{\frac{s}{q}}. \tag{4.3}$$

Also,

$$\int_{\sigma(s)}^t c(\tau)\Delta\tau = (q-1) \sum_{\tau\in[\sigma(s),t)_{q^{\mathbb{N}_0}}} \frac{1}{\tau^{\frac{1}{2}}} \geq \frac{1}{(sq)^{\frac{1}{2}}}. \tag{4.4}$$

Therefore, using (4.3) and (4.4) gives us

$$\begin{aligned} \int_1^t a(s)\left(\int_{\sigma(s)}^t c(\tau)\Delta\tau\right)\left(\int_1^s b(\nu)\Delta\nu\right)\Delta s &\geq \frac{1}{q}\int_1^t a(s)\Delta s \qquad (4.5)\\ &= \frac{q^{\frac{1}{6}}-1}{(1+\sqrt{q})q^{\frac{8}{3}}} \sum_{\tau\in[1,t)_{q^{\mathbb{N}_0}}} \frac{1}{\tau^{\frac{2}{3}}} \cdot \tau. \end{aligned}$$

Thus, taking the limit of (4.5) as $t \to \infty$, *we have that Equation (3.22) holds. Hence, every nonoscillatory solution of system (4.2) belongs to Type (b). Moreover,* $\left(\frac{1}{t^{1/6}}, -\frac{(1+\sqrt{q})q^{3/2}}{\sqrt{t}}, \frac{q}{t}\right)$ *is such a solution of system (4.2).*

The following theorem shows the existence criteria for nonoscillatory solutions of system (1.1) and we need the monotonicity conditions on f, g, and h.

Theorem 4.3 *If* $f, g, h : \mathbb{R} \to \mathbb{R}$ *are nondecreasing continuous functions and*

$$I_a < \infty,\ I_b < \infty,\ \textit{and}\ I_c < \infty, \tag{4.6}$$

then system (1.1) has a nonoscillatory solution.

Proof Let (4.6) hold. Then there exists $t_1 \in \mathbb{T}$ such that

$$\int_{t_1}^{\infty} a(r) f\left(1+\int_{t_1}^{r} b(s) g\left(1+h(2) \int_{s}^{\infty} c(\tau) \Delta \tau\right) \Delta s\right) \Delta r<1.$$

Let $\mathcal{B}$ be the Banach space consisting of the bounded and continuous functions on $\mathbb{T}$ with $\|x\|=\sup_{t \geq t_1, t \in \mathbb{T}}|x(t)|$ and the point-wise ordering $\leq$. Let $\mathcal{S}$ be a subset of $\mathcal{B}$ such that

$$\mathcal{S}=\left\{x \in \mathcal{B}: 1 \leq x(t) \leq 2,\ t \in\left[t_1, \infty\right)_T\right\}.$$

One can easily see that $\inf \mathcal{Q} \in \mathcal{S}$ and $\sup \mathcal{Q} \in \mathcal{S}$ for any subset $\mathcal{Q}$ of $\mathcal{S}$. Define an operator $L: \mathcal{S} \rightarrow \mathcal{B}$ such that

$$(L x)(t)=1+\int_{t_1}^{t} a(r) f\left(1+\int_{t_1}^{r} b(s) g\left(1+\int_{s}^{\infty} c(\tau) h(x(\tau)) \Delta \tau\right) \Delta s\right) \Delta r,$$

where $t \in\left[t_1, \infty\right)_T$. Since $x \in \mathcal{S}$ and the fact that f, g and h are nondecreasing, $(L x)(t) \geq 1$ for all $t \in\left[t_1, \infty\right)_T$, and

$$(L x)(t) \leq 1+\int_{t_1}^{t} a(r) f\left(1+\int_{t_1}^{r} b(s) g\left(1+\int_{s}^{\infty} c(\tau) h(2) \Delta \tau\right) \Delta s\right) \Delta r \leq 2.$$

The positivity of a, b, and c and the monotonicity of f, g, and h ensure that Lx is an increasing mapping into itself, i.e. $L x: \mathcal{S} \rightarrow \mathcal{S}$. Then by the Knaster's fixed-point theorem [9], one can have that there does exist $x \in \mathcal{S}$ such that $L x=x$. Therefore, if we set

$$z(t)=1+\int_{t}^{\infty} c(\tau) h(x(\tau)) \Delta \tau \text { and } \ y(t)=1+\int_{t_1}^{t} b(s) g(z(s)) \Delta s,$$

then we obtain that (x, y, z) is a nonoscillatory solution of (1.1), as desired. □

Note that eliminating the nonoscillatory solutions of system (1.1) except for Type (b) requires the product condition on α, β and γ when the sufficient conditions are triple integrals, see Theorems 3.2, 3.3, and 3.5. On the other hand, it is not necessary for the single and double integrals, see Theorems 2.1 and 2.2. Here, an interesting question arises whether there is a system for which single and double integrals are inconclusive while the triple integrals are conclusive.

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