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OSCILLATION CRITERIA FOR SECOND ORDER STRONGLY SUPERLINEAR AND STRONGLY SUBLINEAR DYNAMIC INCLUSIONS

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Abstract. In this paper, we establish some oscillation criteria for strongly superlinear and strongly sublinear dynamic inclusions. Oscillation problems are in differential and difference equations have become very attractive recently. These areas have started to be unified and extended for more powerful general theory, so called dynamic equations on time scales. Results in this paper even are new in continuous case.

Keywords. Oscillation; Dynamic inclusions; Time scales; Superlinear Inclusions, Sublinear inclusions.

AMS (MOS) subject classification: 34N05, 34K11, 34A60.

1 Introduction

This paper is concerned with some oscillatory behavior of solutions of second-order nonlinear dynamic inclusions of the form

$$(p(t)x^{\Delta}(t))^{\Delta} \in F(t, x^{\sigma}(t)) \quad \text{a.e.} \quad t \geq t_0, \tag{1}$$

subject to the following hypotheses:

(H1) $p \in C_{rd}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$ such that

$$A(t) := \int_t^{\infty} \frac{\Delta s}{p(s)} < \infty, \quad t \geq t_0.$$

(H2) $F : [t_0,\infty)_{\mathbb{T}} \times \mathbb{R} \to 2^{\mathbb{R}} \setminus \emptyset$ is a multifunction with compact and convex values such that $|F(., u)| = \sup\{|y| : y \in F(t, u)\}$ and $F(t, u) > 0$ means $y > 0$ for each $y \in F(t, u)$.

(H3)

$$F(t,u) < 0 \quad \text{for } (t,u) \in [t_0,\infty)_{\mathbb{T}} \times \mathbb{R}^+,$$

$$F(t,u) > 0 \quad \text{for } (t,u) \in [t_0,\infty)_{\mathbb{T}} \times \mathbb{R}^-.$$

Dynamic inclusions represent an important generalization of dynamic equations. The solution to a dynamic inclusion is a reachable set, instead of a single trajectory. The solving procedure for dynamic inclusions is quite complicated compared to the numerical methods for dynamic equations. The algorithm is based on some concepts of the optimal control theory. Dynamic inclusions arise in many situations including dynamic variational inequalities, projected dynamical systems, dynamic Coulomb friction problems and fuzzy set arithmetic. For example, the basic rule for Coulomb friction is that the friction force has magnitude μN in the direction opposite to the direction of slip, where N is the normal force and μ is a constant (the friction coefficient). However, if the slip is zero, the friction force can be any force in the correct plane with magnitude smaller than or equal to μN. Thus, writing the friction force as a function of position and velocity leads to a set-valued function.

Oscillation problems in discrete and continuous cases have become very popular recently. These areas have started to be unified and extended under much more powerful general theory, so called time scales which are nonempty subset of real numbers and denoted by $\mathbb{T}$. The theory was initiated by Hilger [9] in 1988. We will mention time scales calculus in the next section briefly. Nevertheless, we recommend two excellent books by Bohner and Peterson [5, 6] for more details.

The main purpose of this paper is to establish some oscillation criteria for dynamic inclusion (1). Grace, Agarwal, and O'Regan [7] establish sufficient conditions for the oscillation for second order differential inclusions

$$(p(t)x'(t))' \in F(t,x(t)) \text{ for } a.e.\ t \geq t_0,$$

only when $A(t) = \infty$. Results in this paper even are new in continuous case. We earlier investigated oscillation criteria for (1) when $A(t) = \infty$ on time scales, see [2]. Our arguments in this paper are related with papers by Grace, Agarwal, Bohner and O'Regan [8], Merdivenci Atıcı and Biles [10], Bohner and Tisdell [3], Akın-Bohner and Sun [1].

Throughout this paper we assume that $\mathbb{T}$ is unbounded above. Let $L^1_{loc}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ be the set of all real-valued locally Δ-integrable functions, that is, the set of Δ- integrable over each compact interval of $[t_0,\infty)_{\mathbb{T}}$. By a solution x of (1), we mean there exists a function $y \in L^1_{loc}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ such that $y(t) \in F(t,x^\sigma(t))$ a.e. on $[t_0,\infty)_{\mathbb{T}}$ and $(p(t)x^\Delta(t))^\Delta = y(t)$ a.e. on $[t_0,\infty)_{\mathbb{T}}$. A solution x of (1) is called oscillatory if it is neither eventually positive nor eventually negative; otherwise it is called nonoscillatory. Inclusion (1) is called oscillatory if all solutions are oscillatory.

2 Preliminary Results

Two most popular examples of time scales are $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. Some other interesting time scales exist, and they give rise to plenty of applications such as the study of population dynamics model (see [8], pages 15 and 71). By a time-scale interval we mean $[t_0, t_1]_{\mathbb{T}} = \{t \in \mathbb{T} : t_0 \leq t \leq t_1\}$, where $t_0, t_1 \in \mathbb{T}$ and other time-scale intervals are defined similarly. We define the forward and backward jump operators σ, $\rho : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}$$

(supplemented by $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$). A point $t \in \mathbb{T}$ with $t > \inf \mathbb{T}$ is called *right-scattered, right-dense, left-scattered* and *left-dense* if $\sigma(t) > t$, $\sigma(t) = t$, $\rho(t) < t$ and $\rho(t) = t$ holds, respectively. Points are left-dense and right-dense at the same time are called *dense*. The set $\mathbb{T}^\kappa$ is derived from $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. *The graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

Hence the graininess function is 0 if $\mathbb{T} = \mathbb{R}$ while it is 1 if $\mathbb{T} = \mathbb{Z}$. Let f be a function defined on $\mathbb{T}$, then we define the delta derivative of f at $t \in \mathbb{T}^\kappa$, denoted by $f^\Delta(t)$, to be the number (provided it exists) with the property such that for every $\epsilon > 0$, there exists a neighborhood U of t with

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t) \left[\sigma(t) - s\right] \right| \leq \epsilon \left|\sigma(t) - s\right| \qquad \text{for all } s \in U.$$

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at each right-dense point and if there exists a finite left-sided limit at all left-dense points. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$. f is said to be (delta) differentiable if its derivative exists.

The derivative and the shift operator σ are related by the formula

$$f^\sigma = f + \mu f^\Delta.$$

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient $\frac{f}{g}$ of two differentiable functions f and g

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f^\sigma(t)g^\Delta(t) = f^\Delta(t)g^\sigma(t) + f(t)g^\Delta(t),$$

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g^\sigma(t)}, \quad \text{where} \quad gg^\sigma \neq 0.$$

For $a, b \in \mathbb{T}$ and a differentiable function f, the Cauchy integral of f^Δ is defined by

$$\int_a^b f^\Delta(t)\Delta t = f(b) - f(a).$$

Other useful formulas are as follows:

$$\int_t^{\sigma(t)} f(s)\Delta s = \mu(t)f(t),$$

$$\int_a^b f(t)g^\Delta(t)\Delta t = f(b)g(b) - f(a)g(a) - \int_a^b f^\Delta(t)g^\sigma(t)\Delta t.$$

The chain rule [8, Theorem 1.90]

$$\frac{(x^{1-\lambda})^\Delta}{1-\lambda} = x^\Delta \int_0^1 [hx^\sigma + (1-h)x]^{-\lambda} dh,$$

where $\lambda > 0$ and x is such that the right-hand side the above inequality is well-defined, is crucial to prove the following lemma, see [5, Lemma 2.1].

Lemma 1. *Suppose* $|x|^\Delta$ *is of one sign on* $[t_0, \infty)_{\mathbb{T}}$ *and* $\lambda > 0$*. Then*

$$\frac{|x|^\Delta}{(|x|^\sigma)^\lambda} \le \frac{(|x|^{1-\lambda})^\Delta}{1-\lambda} \le \frac{|x|^\Delta}{|x|^\lambda} \quad on \quad [t_0, \infty)_{\mathbb{T}}.$$

3 Main Results

In this section, we investigate oscillatory behavior of dynamic inclusion (1). According to the following classification, we will obtain oscillation criteria for strongly superlinear and strongly sublinear dynamic inclusions.

Definition 1. *Inclusion* (1) *(or* F*) is said to be* **strongly superlinear** *if there exist a function* $f : [t_0, \infty)_{\mathbb{T}} \times \mathbb{R} \mapsto \mathbb{R}$ *and a constant* $\beta > 1$ *such that* $uf(t, u) > 0$ *for a.e.* $t \ge t_0$, $u \ne 0$ *satisfying*

$$|F(t, u)| \ge f(t, u) \quad \text{for } (t, u) \in [t_0, \infty)_{\mathbb{T}} \times \mathbb{R}^+$$

$$|F(t, u)| \ge -f(t, u) \quad \text{for } (t, u) \in [t_0, \infty)_{\mathbb{T}} \times \mathbb{R}^-$$

and

$$\frac{|f(t, x)|}{|x|^\beta} \le \frac{|f(t, y)|}{|y|^\beta} \quad \text{for } |x| \le |y|,\ xy > 0, \quad \text{a.e. } t \ge t_0, \tag{2}$$

and it is said to be **strongly sublinear** *if there exist a function* $f : [t_0, \infty)_{\mathbb{T}} \times \mathbb{R} \mapsto \mathbb{R}$ *and a constant* $0 < \gamma < 1$ *such that* $uf(t, u) > 0$ *for a.e.* $t \ge t_0$, $u \ne 0$ *satisfying*

$$|F(t, u)| \ge f(t, u) \quad \text{for } (t, u) \in [t_0, \infty)_{\mathbb{T}} \times \mathbb{R}^+$$

$$|F(t, u)| \ge -f(t, u) \quad \text{for } (t, u) \in [t_0, \infty)_{\mathbb{T}} \times \mathbb{R}^-$$

and

$$\frac{|f(t, x)|}{|x|^\gamma} \ge \frac{|f(t, y)|}{|y|^\gamma} \quad \text{for } |x| \le |y|,\ xy > 0, \quad \text{a.e. } t \ge t_0. \tag{3}$$

If (2) holds when $\beta = 1$, then (1) is called superlinear while if (3) holds when $\gamma = 1$, (1) is called sublinear.

The following lemma is very crucial and will be used several times.

Lemma 2. *Suppose x solves (1) and is of one sign on $[t_0, \infty)_{\mathbb{T}}$. Then either*

$$xx^{\Delta} \geq 0 \quad on \quad [t_0, \infty)_{\mathbb{T}} \tag{4}$$

or there exists $t_1 \geq t_0$, $t_1 \in \mathbb{T}$ such that

$$xx^{\Delta} \leq 0 \quad on \quad [t_1, \infty)_{\mathbb{T}}. \tag{5}$$

Moreover, assume that there exists a function $f : [t_0, \infty)_{\mathbb{T}} \times \mathbb{R} \mapsto \mathbb{R}$ such that $uf(t, u) > 0$ for a.e. $t \geq t_0$ and let

$$\bar{c} = \{|x(t_0)| + p(t_0)|x^{\Delta}(t_0)|A(t_0)\} sgn x(t_0)$$

and

$$\hat{c} = \begin{cases} \frac{x(t_0)}{A(t_0)} & if\ (4)\ holds \\ p(t_1)x^{\Delta}(t_1) sgn x(t_0) & if\ (5)\ holds\ . \end{cases}$$

Then

$$|x| \leq |\bar{c}| \quad on \quad [t_0, \infty)_{\mathbb{T}}, \ where \ \ \bar{c}x > 0 \tag{6}$$

and

$$|x| \geq |\hat{c}A| \quad on \quad [t_0, \infty)_{\mathbb{T}}, \ where \ \ \hat{c}Ax > 0. \tag{7}$$

Proof. Assume that x solves (1) and $x > 0$ on $[t_0, \infty)_{\mathbb{T}}$. The case $x < 0$ can be shown similarly. If (4) does not hold, then there exits $t_1 > t_0$, $t_1 \in \mathbb{T}$ such that $x^{\Delta}(t_1) < 0$. From (H2), we have

$$(p(t)x^{\Delta}(t))^{\Delta} < 0 \text{ for } t \geq t_0 \tag{8}$$

and so

$$p(t)x^{\Delta}(t) \leq p(t_1)x^{\Delta}(t_1) < 0 \qquad \text{for } t \geq t_1.$$

Therefore, $x^{\Delta}(t) < 0$ for $t \geq t_1$. This proves (5).

Moreover, (8) also implies

$$p(t)x^{\Delta}(t) \leq p(t_0)x^{\Delta}(t_0), \quad t \geq t_0$$

or

$$x^{\Delta}(t) \leq \frac{p(t_0)x^{\Delta}(t_0)}{p(t)}, \quad t \geq t_0.$$

Integrating the above inequality from t_0 to t, $t \geq t_0$ we find

$$x(t) \leq x(t_0) + p(t_0)x^{\Delta}(t_0) \int_{t_0}^{t} \frac{\Delta s}{p(s)},$$

and so

$$\begin{aligned} |x(t)| &\leq |x(t_0)| + p(t_0)|x^\Delta(t_0)| \int_{t_0}^t \frac{\Delta s}{p(s)} \\ &\leq |x(t_0)| + p(t_0)|x^\Delta(t_0)|A(t_0). \end{aligned} \tag{9}$$

Denote

$$\bar{c} := x(t_0) + p(t_0)|x^\Delta(t_0)|A(t_0). \tag{10}$$

This proves (6).

In order to show (7) we consider two cases: If (4) holds, then

$$x(t) \geq x(t_0) = \hat{c}A(t_0) \geq \hat{c}A(t) \quad \text{for all } t \geq t_0.$$

If (5) holds, then let

$$y(t) = (p(t)x^\Delta(t))^\Delta \tag{11}$$

and $y(t) \in F(t, x^\sigma(t))$, $y \in L^1_{loc}([t_0,\infty)_\mathbb{T}, \mathbb{R})$. Integrating (11) from v to s, $v, s \geq t_0$, $v, s \in \mathbb{T}$ we have

$$p(s)x^\Delta(s) = p(v)x^\Delta(v) + \int_v^s y(\tau)\Delta\tau$$

and so

$$x^\Delta(s) = \frac{p(v)x^\Delta(v)}{p(s)} + \frac{1}{p(s)} \int_v^s y(\tau)\Delta\tau. \tag{12}$$

Note that

$$-y(t) \geq f(t, x^\sigma(t)) \text{ for } (t,x) \in [t_0,\infty)_\mathbb{T} \times \mathbb{R}^+. \tag{13}$$

Taking (13) into account in (12) and integrating (12) from t to $u \in \mathbb{T}$, $u \geq t_0$ we have

$$x(u) - x(t) \leq p(v)x^\Delta(v) \int_t^u \frac{\Delta s}{p(s)} - \int_t^u \frac{1}{p(s)} \int_v^s f(\tau, x^\sigma(\tau))\Delta\tau\Delta s \tag{14}$$

or

$$x(t) \geq -p(v)x^\Delta(v) \int_t^u \frac{\Delta s}{p(s)} + \int_t^u \frac{1}{p(s)} \int_v^s f(\tau, x^\sigma(\tau))\Delta\tau\Delta s.$$

Letting $v = t_1$ and $u \to \infty$ yields

$$\begin{aligned} x(t) &\geq -p(t_1)x^\Delta(t_1) \int_t^\infty \frac{\Delta s}{p(s)} + \int_t^\infty \frac{1}{p(s)} \int_{t_1}^s f(\tau, x^\sigma(\tau))\Delta\tau\Delta s \\ &\geq -p(t_1)x^\Delta(t_1)A(t) \\ &:= \hat{c}A(t), \end{aligned}$$

which completes the proof. □

In the following theorem we do not need to assume that (1) is strongly superlinear or strongly sublinear.

Theorem 3. *In addition to conditions (H1)-(H3), assume*

(H4) there exists a function $f : [t_0, \infty)_{\mathbb{T}} \times \mathbb{R} \mapsto \mathbb{R}$ *satisfying*

$$f(t, x) \leq f(t, y), \quad x \leq y, \quad t \geq t_0.$$

If

$$\int_{t_0}^{\infty} \frac{1}{p(s)} \int_{t_0}^{s} |f(\tau, cA^{\sigma}(\tau))| \Delta\tau \Delta s = \infty, \tag{15}$$

for every nonzero constant c, then (1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1) such that $x > 0$ on $[t_0, \infty)_{\mathbb{T}}$. The case $x < 0$ can be shown similarly. From (7) and (H4), we obtain

$$f(\tau, x^{\sigma}(\tau)) \geq f(\tau, \hat{c}A^{\sigma}(\tau)) \text{ for all } \tau \geq t_0. \tag{16}$$

Using (14) with $u = v = t_1 \leq t$ and (16), we have

$$\begin{aligned} x(t) \leq & \quad x(t_1) + p(t_1)x^{\Delta}(t_1) \textstyle\int_{t_1}^{t} \frac{\Delta s}{p(s)} - \int_{t_1}^{t} \frac{1}{p(s)} \int_{t_1}^{s} f(\tau, x^{\sigma}(\tau))\Delta\tau\Delta s & (17) \\ \leq & \quad x(t_1) + p(t_1)x^{\Delta}(t_1) \textstyle\int_{t_1}^{t} \frac{\Delta s}{p(s)} - \int_{t_1}^{t} \frac{1}{p(s)} \int_{t_1}^{s} f(\tau, \hat{c}A^{\sigma}(\tau))\Delta\tau\Delta s, & (18) \end{aligned}$$

which contradicts to the positivity of x by (15) as $t \to \infty$. □

The following result is concerned with oscillatory behavior of inclusion (1) when F is strongly superlinear, see (2).

Theorem 4. *In addition to conditions (H1)-(H3), assume* (1) *is strongly superlinear. If*

$$\int_{t_0}^{\infty} |f(\tau, cA^{\sigma}(\tau)| \Delta\tau = \infty \tag{19}$$

for every nonzero constant c, then inclusion (1) *is oscillatory.*

Proof. Assume x is a nonoscillatory solution of (1) such that $x > 0$ on $[t_0, \infty)_{\mathbb{T}}$. The case $x < 0$ can be shown similarly. Since (1) is strongly superlinear and (7) holds, there exist a function $f : [t_0, \infty)_{\mathbb{T}} \times \mathbb{R} \mapsto \mathbb{R}$ and a constant $\beta > 1$ such that

$$\frac{f(\tau, x^{\sigma}(\tau)}{(x^{\sigma}(\tau))^{\beta}} \geq \frac{f(\tau, \hat{c}A^{\sigma}(\tau))}{(\hat{c}A^{\sigma}(\tau))^{\beta}} \quad \text{for all } t \geq t_1 \geq t_0. \tag{20}$$

Using (14) with $t \geq u \geq t_1 = v$, we obtain

$$\begin{aligned} x(u) \quad \geq & \quad -p(t_1)x^{\Delta}(t_1) \int_{u}^{t} \frac{\Delta s}{p(s)} + \int_{u}^{t} \frac{1}{p(s)} \int_{t_1}^{u} f(\tau, x^{\sigma}(\tau))\Delta\tau\Delta s & (21) \\ \geq & \quad bA(u) + A(u) \int_{t_1}^{u} f(\tau, x^{\sigma}(\tau))\Delta\tau & (22) \\ \geq & \quad bA(u) + A(u) \int_{t_1}^{u} \frac{f(\tau, \hat{c}A^{\sigma}(\tau))}{(\hat{c}A^{\sigma}(\tau))^{\beta}} (x^{\sigma}(\tau))^{\beta} \Delta\tau, & (23) \end{aligned}$$

where $b := p(t_1)|x^\Delta(t_1)|$ and we use (20) and so

$$\frac{x(u)}{A(u)} \geq b + (\hat{c})^{-\beta} \int_{t_1}^{u} f(\tau, \hat{c}A^\sigma(\tau)) \left(\frac{x^\sigma(\tau)}{A^\sigma(\tau)}\right)^\beta \Delta\tau := w(u).$$

By Lemma 2,

$$w^\Delta(\tau) = (\hat{c})^{-\beta} f(\tau, \hat{c}A^\sigma(\tau)) \left(\frac{x^\sigma(\tau)}{A^\sigma(\tau)}\right)^\beta \geq (\hat{c})^{-\beta} f(\tau, \hat{c}A^\sigma(\tau)) \left(w^\sigma(\tau)\right)^\beta$$

and therefore we have

$$-\left(w^{1-\beta}\right)^\Delta(\tau) \geq \frac{\beta-1}{(\hat{c})^\beta} f(\tau, \hat{c}A^\sigma(\tau)).$$

Integrating the above inequality from t_1 to $t \geq t_1$, we obtain

$$\left(w^{1-\beta}\right)(t_1) \geq \frac{\beta-1}{(\hat{c})^\beta} \int_{t_1}^{t} f(\tau, \hat{c}A^\sigma(\tau))\Delta\tau,$$

which contradicts with the way b is chosen as $t \to \infty$. □

The following result is concerned with oscillatory behavior of inclusion (1) when F is strongly sublinear, see (3).

Theorem 5. *In addition to (H1)-(H4), assume that* (1) *is strongly sublinear. If*

$$\int_{t_0}^{\infty} \frac{1}{p(s)} \int_{t_0}^{s} |f(\tau, c)|\Delta\tau\Delta s = \infty \tag{24}$$

for every nonzero constant c, then inclusion (1) *is oscillatory.*

Proof. Assume x is a nonoscillatory solution of (1) such that $x > 0$ on $[t_0, \infty)_{\mathbb{T}}$. The case $x < 0$ can be shown similarly. By Lemma 2, either (4) or (5) holds. If (4) holds, then we have

$$x^\sigma(t) \geq x(t) \geq x(t_0) \text{ for all } t \geq t_0$$

and so by (14) with $u = v = t_0 \leq$ and (H4)

$$x(t) \leq x(t_0) + p(t_0)x^\Delta(t_0) \int_{t_0}^{t} \frac{1}{p(s)} - \int_{t_0}^{t} \frac{1}{p(s)} \int_{t_0}^{s} f(\tau, x(t_0))\Delta\tau\Delta s,$$

which is a contradiction to the positivity of x as $t \to \infty$. If (5) holds, then by (6) and (3) with $0 < \gamma < 1$ we have

$$\frac{f(\tau, x^\sigma(\tau))}{(x^\sigma(\tau))^\gamma} \geq \frac{f(\tau, \bar{c})}{\bar{c}^\gamma} \text{ for all } t \geq t_1. \tag{25}$$

By (12) with $v = t_1 \leq t$, (13), and (25), we have

$$\begin{aligned} x^\Delta(t) &= \frac{p(t_1)x^\Delta(t_1)}{p(t)} + \frac{1}{p(t)}\int_{t_1}^t y(\tau)\Delta\tau \\ &\leq -\frac{1}{p(t)}\int_{t_1}^t f(\tau, x^\sigma(\tau))\Delta\tau \\ &\leq -\frac{1}{p(t)}\int_{t_1}^t \frac{(x^\sigma(\tau))^\gamma f(\tau,\bar{c})}{(\bar{c})^\gamma}\Delta\tau \\ &\leq -\frac{(\bar{c})^{-\gamma}}{p(t)} x^\gamma(t)\int_{t_1}^t f(\tau,\bar{c})\Delta\tau. \end{aligned}$$

Now from Lemma 1

$$\frac{(\bar{c})^{-\gamma}}{p(t)}\int_{t_1}^t f(\tau,\bar{c})\Delta\tau \leq -\frac{x^\Delta(t)}{x^\gamma(t)} \leq -\frac{(x^{1-\gamma})^\Delta(t)}{1-\gamma}.$$

Integrating the above inequality from t_1 to $t \geq t_1$, we obtain

$$\begin{aligned} x^{1-\gamma}(t_1) &\geq x^{1-\gamma}(t) + \frac{1-\gamma}{(\bar{c})^\gamma}\int_{t_1}^t \frac{1}{p(s)}\int_{t_1}^s f(\tau,\bar{c})\Delta\tau\Delta s \\ &\geq \frac{1-\gamma}{(\bar{c})^\gamma}\int_{t_1}^t \frac{1}{p(s)}\int_{t_1}^s f(\tau,\bar{c})\Delta\tau\Delta s, \end{aligned}$$

which is contradiction as $t \to \infty$ by (24). □

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