Dynamic Equations on Time Scales

An Introduction with Applications

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Preface

> On becoming familiar with difference equations and their close relation to differential equations, I was in hopes that the theory of difference equations could be brought completely abreast with that for ordinary differential equations.
>
> [Hugh L. Turrittin, "My Mathematical Expectations", Springer Lecture Notes 312 (page 10), 1973]

> A major task of mathematics today is to harmonize the continuous and the discrete, to include them in one comprehensive mathematics, and to eliminate obscurity from both.
>
> [E. T. Bell, "Men of Mathematics", Simon and Schuster, New York (page 13/14), 1937]

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his PhD thesis [**159**] in 1988 (supervised by Bernd Aulbach) in order to unify continuous and discrete analysis. This book is an introduction to the study of dynamic equations on time scales. Many results concerning differential equations carry over quite easily to corresponding results for difference equations, while other results seem to be completely different in nature from their continuous counterparts. The study of dynamic equations on time scales reveals such discrepancies, and helps avoid proving results twice, once for differential equations and once for difference equations. The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a so-called time scale, which is an arbitrary closed subset of the reals. By choosing the time scale to be the set of real numbers, the general result yields a result concerning an ordinary differential equation as studied in a first course in differential equations, and by choosing the time scale to be the set of integers, the same general result yields a result for difference equations. However, since there are many other time scales than just the set of real numbers or the set of integers, one has a much more general result. We may summarize the above and state that

Unification and **Extension**

are the two *main features* of the time scales calculus.

The time scales calculus has a tremendous potential for applications. For example, it can model insect populations that are continuous while in season (and may follow a difference scheme with variable step-size), die out in (say) winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population.

The *audience* for this book is as follows:

1. Most parts of this book are appropriate for students who have had a first course in both calculus and linear algebra. Usually, a first course in differential equations does not even consider the discrete case, which students encounter in numerous applications, for example in biology, engineering, economics, physics, neural networks, social sciences and so on. A course taught out of this book would simultaneously teach the continuous and the discrete theory, which would better prepare students for these applications. The first four chapters can be used for an introductory course on time scales. They contain plenty of exercises, and we included many of the solutions at the end of the book. Altogether, this book contains 210 exercises, many of them consisting of several separate parts.
2. The last four chapters can be used for an advanced course on time scales at the beginning graduate level. Also, a special topics course is appropriate. These chapters also contain many exercises, however, most of their answers are not included in the solutions section at the end of the book.
3. A third group of audience for this book might be graduate students who are interested in the subject for a thesis project at the masters or doctoral level. Some of the exercises describe open problems that can be used as a starting point for such a project. The "Notes and References" sections at the end of each chapter also point out directions of further possible research.
4. Finally, researchers with a knowledge of differential or difference equations, who want a rather complete introduction into the time scales calculus without going through all the current literature, may also find this book very useful.

Most of the results given in this book have recently been investigated by Stefan Hilger and by the authors of this book (together with their research collaborators R. P. Agarwal, C. Ahlbrandt, E. Akın, S. Clark, O. Došlý, P. Eloe, L. Erbe, B. Kaymakçalan, D. Lutz, and R. Mathsen). Other results presented or results related to the presented ones have been obtained by D. Anderson, F. Atıcı, B. Aulbach, J. Davis, G. Guseinov, J. Henderson, R. Hilscher, S. Keller, V. Lakshmikantham, C. Pötzsche, Z. Pospíšil, S. Siegmund, and S. Sivasundaram. Many of these results are presented in this book at a level that will be easy for an undergraduate mathematics student to understand.

In Chapter 1 the calculus on time scales as developed in [**160**] by Stefan Hilger is introduced. A time scale $\mathbb{T}$ is an arbitrary closed subset of the reals. For functions $f : \mathbb{T} \to \mathbb{R}$ we introduce a derivative and an integral. Fundamental results, e.g., the product rule and the quotient rule, are presented. Further results concerning differentiability and integrability, which have been previously unpublished but are easy to derive, are given as they are needed in the remaining parts of the book. Important examples of time scales, which we will consider frequently throughout the book, are given in this chapter. Such examples contain of course $\mathbb{R}$ (the set of all real numbers, which gives rise to differential equations) and $\mathbb{Z}$ (the set of all integers, which gives rise to difference equations), but also the set of all integer multiples of a number $h > 0$ and the set of all integer powers of a number $q > 1$, including 0 (this time scale gives rise to so-called q-difference equations, see e.g., [**58, 247, 253**]). Other examples are sets of disjoint closed intervals (which have applications e.g., in population dynamics) or even "exotic" time scales such as the Cantor set. After discussing these examples, we also derive analogues of the chain rule. Taylor's formula is presented, which is helpful in the study of boundary value problems.

In Chapter 2 we introduce the Hilger complex plane, following closely Stefan Hilger's paper [**164**]. We use the so-called cylinder transformation to introduce the exponential function on time scales. This exponential function is then shown to satisfy an initial value problem involving a first order linear dynamic equation. We derive many properties of the exponential function and use it to solve all initial value problems involving first order linear dynamic equations. For the nonhomogeneous cases we utilize a variation of constants technique.

Next, we consider second order linear dynamic equations in Chapter 3. Again, there are several kinds of second order linear homogeneous equations, which we solve (in the constant coefficient case) using hyperbolic and trigonometric functions. Wronskian determinants are introduced and Abel's theorem is used to develop a reduction of order technique to find a second solution in case one solution is already known. Certain dynamic equations of second order with nonconstant coefficients (e.g., the Euler–Cauchy equation) are also considered. We also present a variation of constants formula that helps in solving nonhomogeneous second order linear dynamic equations. The Laplace transformation on a general time scale is introduced and many of its properties are derived.

Next, in Chapter 4, we study self-adjoint dynamic equations on time scales. Such equations have been well studied in the continuous case (where they are also called Sturm–Liouville equations) and in the discrete case (where they are called Sturm–Liouville difference equations). In this chapter we only consider such equations of second order. We investigate disconjugacy of self-adjoint equations and use the corresponding Green's function to study boundary value problems. Also the theory of Riccati equations is developed in the general setting of time scales, and we present a characterization of disconjugacy in terms of a certain quadratic functional. An analogue of the classical Prüfer transformation, which has proved to be a useful tool in oscillation theory of Sturm–Liouville equations, is given as well. In the last section, we examine eigenvalue problems on time scales. For the case of separated boundary conditions we present an oscillation result on the number of zeros of the kth eigenfunction. Such a result goes back to Sturm in the continuous case, and its discrete counterpart is contained in the book by Kelley and Peterson [**191**, Theorem 7.6]. Further results on eigenvalue problems contain a comparison theorem and Rayleigh's principle.

Chapter 5 is concerned with linear systems of dynamic equations on a time scale. Uniqueness and existence theorems are presented, and the matrix exponential on a time scale is introduced. We also examine fundamental systems and their Wronskian determinants and give a variation of constants formula. The case of constant coefficient matrices is also investigated, and a Putzer algorithm from [**31**] is presented. This chapter contains a section on self-adjoint vector equations. Such equations are a special case of symplectic systems as discussed in Chapter 7. They are closely connected to certain matrix Riccati equations, and in this section we also discuss oscillation results for those systems. Further results contain a discussion on asymptotic behavior of solutions of linear systems of dynamic equations. Related results are time scales versions of Levinson's perturbation lemma and the Hartman–Wintner theorem. Finally we study higher order linear dynamic equations on a time scale. We give conditions that imply that corresponding initial value problems have unique solutions. Abel's formula is given, and the notion of a generalized zero of a solution is introduced.

Chapter 6 is concerned with dynamic inequalities on time scales. Analogues of Gronwall's inequality, Hölder's inequality, and Jensen's inequality are presented. We also derive Opial's inequality and point out its applications in the study of initial or boundary value problems. Opial inequalities have proved to be a useful tool in differential equations and in difference equations, and in fact there is an entire book [**19**] devoted to them. Next, we prove Lyapunov's inequality for Sturm–Liouville equations of second order with positive coefficients. It can be used to give sufficient conditions for disconjugacy. We also offer an extension of Lyapunov's inequality to the case of linear Hamiltonian dynamic systems. Further results in this section concern upper and lower solutions of boundary value problems and are contained in the article [**32**] by Akın.

In Chapter 7 we consider linear symplectic dynamic systems on time scales. This is a very general class of systems that contains for example linear Hamiltonian dynamic systems which in turn contain Sturm–Liouville dynamic equations of higher order (and hence of course also of order two) and self-adjoint vector dynamic equations. We derive a Wronskian identity for such systems as well as a close connection to certain matrix Riccati dynamic equations. Disconjugacy of symplectic systems is introduced as well. Some of the results in this chapter are due to Roman Hilscher, who considered Hamiltonian systems on time scales in [**166, 167, 168**]. Other results are contained in a paper by Ondřej Došlý and Roman Hilscher [**121**].

Chapter 8 contains several possible extensions of the time scales calculus. In the first section we present an introduction to the concept of measure chains as introduced by Stefan Hilger in [**159**]. The second section contains the proofs of the main local and global existence theorems, that are needed throughout the book. Another extension, which is considered in this last chapter, concerns alpha derivatives. The time scales calculus as presented in this book is a special case of this concept.

Parts or earlier versions of this book have been proofread by D. Anderson, R. Avery, R. Chiquet, C. J. Chyan, L. Cole, J. Davis, L. Erbe, K. Fick, J. Henderson, J. Hoffacker, K. Howard, N. Hummel, B. Karna, R. Mathsen, K. Messer, A. Rosidian, P. Singh, and W. Yin. In particular, we would like to thank Elvan Akın, Roman Hilscher, Billûr Kaymakçalan, and Stefan Siegmund for proofreading the entire manuscript. Finally, we wish to express our appreciation to "Birkhäuser Boston", in particular to Ann Kostant, for their accomplished handling of this manuscript.

Martin Bohner and Allan Peterson

CHAPTER 1

The Time Scales Calculus

1.1. Basic Definitions

A *time scale* (which is a special case of a *measure chain*, see Chapter 8) is an arbitrary nonempty closed subset of the real numbers. Thus

$$\mathbb{R}, \quad \mathbb{Z}, \quad \mathbb{N}, \quad \mathbb{N}_0,$$

i.e., the real numbers, the integers, the natural numbers, and the nonnegative integers are examples of time scales, as are

$$[0,1] \cup [2,3], \quad [0,1] \cup \mathbb{N}, \quad \text{and the Cantor set,}$$

while

$$\mathbb{Q}, \quad \mathbb{R} \setminus \mathbb{Q}, \quad \mathbb{C}, \quad (0,1),$$

i.e., the rational numbers, the irrational numbers, the complex numbers, and the open interval between 0 and 1, are *not* time scales. Throughout this book we will denote a time scale by the symbol $\mathbb{T}$. We assume throughout that a time scale $\mathbb{T}$ has the topology that it inherits from the real numbers with the standard topology.

The calculus of time scales was initiated by Stefan Hilger in his PhD thesis [**159**] in order to create a theory that can unify discrete and continuous analysis. Indeed, below we will introduce the delta derivative f^Δ for a function f defined on $\mathbb{T}$, and it turns out that

(i) $f^\Delta = f'$ is the usual derivative if $\mathbb{T} = \mathbb{R}$ and
(ii) $f^\Delta = \Delta f$ is the usual forward difference operator if $\mathbb{T} = \mathbb{Z}$.

In this section we introduce the basic notions connected to time scales and differentiability of functions on them, and we offer the above two cases as examples. However, the general theory is of course applicable to many more time scales $\mathbb{T}$, and we will give some examples of such time scales in Section 1.3 below and many more examples throughout the rest of this book. Let us start by defining the forward and backward jump operators.

Definition 1.1. Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : \ s > t\},$$

while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

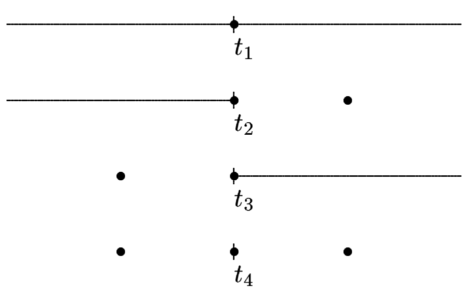
$$\rho(t) := \sup\{s \in \mathbb{T} : \ s < t\}.$$

In this definition we put $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t), where $\emptyset$ denotes the empty set. If $\sigma(t) > t$, we say that t is *right-scattered*, while if $\rho(t) < t$ we say that t is *left-scattered*. Points that are right-scattered and left-scattered at the same time

Table 1.1. Classification of Points

t right-scattered	$\sigma(t) > t$
t right-dense	$\sigma(t) = t$
t left-scattered	$\rho(t) < t$
t left-dense	$\rho(t) = t$

Figure 1.1. Classifications of Points



t_1 is left-dense and right-dense

t_2 is left-dense and right-scattered

t_3 is left-scattered and right-dense

t_4 is left-scattered and right-scattered

(t_4 is isolated)

are called *isolated*. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense*, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called *left-dense*. Points that are right-dense and left-dense at the same time are called *dense*. Finally, the *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

See Table 1.1 for a classification and Figure 1.1 for a schematic classification of points in $\mathbb{T}$. Note that in the definition above both $\sigma(t)$ and $\rho(t)$ are in $\mathbb{T}$ when $t \in \mathbb{T}$. This is because of our assumption that $\mathbb{T}$ is a closed subset of $\mathbb{R}$. We also need below the set $\mathbb{T}^\kappa$ which is derived from the time scale $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. In summary,

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if} \quad \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{if} \quad \sup \mathbb{T} = \infty. \end{cases}$$

Finally, if $f : \mathbb{T} \to \mathbb{R}$ is a function, then we define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ by

$$f^\sigma(t) = f(\sigma(t)) \quad \text{for all} \quad t \in \mathbb{T},$$

i.e., $f^\sigma = f \circ \sigma$.

Example 1.2. Let us briefly consider the two examples $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

(i) If $\mathbb{T} = \mathbb{R}$, then we have for any $t \in \mathbb{R}$

$$\sigma(t) = \inf\{s \in \mathbb{R} : \ s > t\} = \inf(t, \infty) = t$$

and similarly $\rho(t) = t$. Hence every point $t \in \mathbb{R}$ is dense The graininess function μ turns out to be

$$\mu(t) \equiv 0 \quad \text{for all} \quad t \in \mathbb{T}.$$

(ii) If $\mathbb{T} = \mathbb{Z}$, then we have for any $t \in \mathbb{Z}$

$$\sigma(t) = \inf\{s \in \mathbb{Z} : \ s > t\} = \inf\{t+1, t+2, t+3, \dots\} = t+1$$

and similarly $\rho(t) = t-1$. Hence every point $t \in \mathbb{Z}$ is isolated. The graininess function μ in this case is

$$\mu(t) \equiv 1 \quad \text{for all} \quad t \in \mathbb{T}.$$

For the two cases discussed above, the graininess function is a constant function. We will see below that the graininess function plays a central rôle in the analysis on time scales. For the general case, a lot of formulae will have some term containing the factor $\mu(t)$. This term is there in case $\mathbb{T} = \mathbb{Z}$ since $\mu(t) \equiv 1$. However, for the case $\mathbb{T} = \mathbb{R}$ this term disappears since $\mu(t) \equiv 0$ in this case. In various cases this fact is the reason for certain differences between the continuous and the discrete case. One of the many examples of this that we will see later is the so-called scalar Riccati equation (see formula (4.18))

$$z^{\Delta} + q(t) + \frac{z^2}{p(t) + \mu(t)z} = 0$$

on a general time scale $\mathbb{T}$. Note that if $\mathbb{T} = \mathbb{R}$, then we get the well-known Riccati differential equation

$$z' + q(t) + \frac{1}{p(t)} z^2 = 0$$

and if $\mathbb{T} = \mathbb{Z}$, then we get the Riccati difference equation (see, e.g., [**191**, Chapter 6])

$$\Delta z + q(t) + \frac{z^2}{p(t) + z} = 0.$$

Of course, for the case of a general time scale, the graininess function might very well be a function of $t \in \mathbb{T}$, as the reader can verify in the next exercise. For more such examples we refer to Section 1.3.

Exercise 1.3. For each of the following time scales $\mathbb{T}$, find σ, ρ, and μ, and classify each point $t \in \mathbb{T}$ as left-dense, left-scattered, right-dense, or right-scattered:

(i) $\mathbb{T} = \{2^n : \ n \in \mathbb{Z}\}$;
(ii) $\mathbb{T} = \left\{\frac{1}{n} : \ n \in \mathbb{N}\right\} \cup \{0\}$;
(iii) $\mathbb{T} = \left\{\frac{n}{2} : \ n \in \mathbb{N}_0\right\}$;
(iv) $\mathbb{T} = \left\{\sqrt{n} : \ n \in \mathbb{N}_0\right\}$;
(v) $\mathbb{T} = \left\{\sqrt[3]{n} : \ n \in \mathbb{N}_0\right\}$.

Exercise 1.4. Give examples of time scales $\mathbb{T}$ and points $t \in \mathbb{T}$ such that the following equations are not true. Also determine the conditions on t under which those equations are true:

(i) $\sigma(\rho(t)) = t$;
(ii) $\rho(\sigma(t)) = t$.

Exercise 1.5. Is $\sigma : \mathbb{T} \to \mathbb{T}$ one-to-one? Is it onto? If it is not onto, determine the range $\sigma(\mathbb{T})$ of σ. How about $\rho : \mathbb{T} \to \mathbb{T}$? This exercise was suggested by Roman Hilscher.

Exercise 1.6. If $\mathbb{T}$ consists of finitely many points, calculate $\sum_{t\in\mathbb{T}} \mu(t)$.

Throughout this book we make the blanket assumption that a and b are points in $\mathbb{T}$. Often we assume $a \leq b$. We then define the interval $[a,b]$ in $\mathbb{T}$ by

$$[a,b] := \{t \in \mathbb{T} : \ a \leq t \leq b\}.$$

Open intervals and half-open intervals etc. are defined accordingly. Note that $[a,b]^\kappa = [a,b]$ if b is left-dense and $[a,b]^\kappa = [a,b) = [a,\rho(b)]$ if b is left-scattered.

Sometimes the following *induction principle* is a useful tool.

Theorem 1.7 (Induction Principle)**.** *Let $t_0 \in \mathbb{T}$ and assume that*

$$\{S(t) : \ t \in [t_0, \infty)\}$$

is a family of statements satisfying:

I. The statement $S(t_0)$ is true.
II. If $t \in [t_0,\infty)$ is right-scattered and $S(t)$ is true, then $S(\sigma(t))$ is also true.
III. If $t \in [t_0,\infty)$ is right-dense and $S(t)$ is true, then there is a neighborhood U of t such that $S(s)$ is true for all $s \in U \cap (t,\infty)$.
IV. If $t \in (t_0,\infty)$ is left-dense and $S(s)$ is true for all $s \in [t_0,t)$, then $S(t)$ is true.

Then $S(t)$ is true for all $t \in [t_0,\infty)$.

Proof. Let

$$S^* := \{t \in [t_0,\infty) : \ S(t) \text{ is not true}\}.$$

We want to show $S^* = \emptyset$. To achieve a contradiction we assume $S^* \neq \emptyset$. But since S^* is closed and nonempty, we have

$$\inf S^* =: t^* \in \mathbb{T}.$$

We claim that $S(t^*)$ is true. If $t^* = t_0$, then $S(t^*)$ is true from (i). If $t^* \neq t_0$ and $\rho(t^*) = t^*$, then $S(t^*)$ is true from (iv). Finally if $\rho(t^*) < t^*$, then $S(t^*)$ is true from (ii). Hence, in any case,

$$t^* \notin S^*.$$

Thus, t^* cannot be right-scattered, and $t^* \neq \max \mathbb{T}$ either. Hence t^* is right-dense. But now (iii) leads to a contradiction. ☐

Remark 1.8. A dual version of the induction principle also holds for a family of statements $S(t)$ for t in an interval of the form $(-\infty, t_0]$. I.e., to show that $S(t)$ is true for all $t \in (-\infty, t_0]$ we have to show that $S(t_0)$ is true, that $S(t)$ is true at a left-scattered t implies $S(\rho(t))$ is true, that $S(t)$ is true at a left-dense t implies $S(r)$ is true for all r in a left neighborhood of t, and that $S(r)$ is true for all $r \in (t, t_0]$ where t is right-dense implies $S(t)$ is true.

Exercise 1.9. Prove Remark 1.8.

1.2. Differentiation

Now we consider a function $f : \mathbb{T} \to \mathbb{R}$ and define the so-called *delta* (or *Hilger*) *derivative* of f at a point $t \in \mathbb{T}^\kappa$.

Definition 1.10. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t (i.e., $U = (t-\delta, t+\delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$\left|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]\right| \le \varepsilon \,|\sigma(t) - s| \quad \text{for all} \quad s \in U.$$

We call $f^\Delta(t)$ the *delta* (or *Hilger*) *derivative* of f at t.

Moreover, we say that f is *delta* (or *Hilger*) *differentiable* (or in short: *differentiable*) on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. The function $f^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ is then called the (delta) derivative of f on $\mathbb{T}^\kappa$.

Exercise 1.11. Prove that the delta derivative is well defined.

Exercise 1.12. Sometimes it is convenient to have $f^\Delta(t)$ also defined at a point $t \in \mathbb{T} \setminus \mathbb{T}^\kappa$. At such a point we use the same definition as given in Definition 1.10. Prove that an $f : \mathbb{T} \to \mathbb{R}$ has any $\alpha \in \mathbb{R}$ as its derivative at points $t \in \mathbb{T} \setminus \mathbb{T}^\kappa$.

Example 1.13.
(i) If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = \alpha$ for all $t \in \mathbb{T}$, where $\alpha \in \mathbb{R}$ is constant, then $f^\Delta(t) \equiv 0$. This is clear because for any $\varepsilon > 0$,
$$|f(\sigma(t)) - f(s) - 0 \cdot [\sigma(t) - s]| = |\alpha - \alpha| = 0 \le \varepsilon|\sigma(t) - s|$$
holds for all $s \in \mathbb{T}$.
(ii) If $f : \mathbb{T} \to \mathbb{R}$ is defined by $f(t) = t$ for all $t \in \mathbb{T}$, then $f^\Delta(t) \equiv 1$. This follows since for any $\varepsilon > 0$,
$$|f(\sigma(t)) - f(s) - 1 \cdot [\sigma(t) - s]| = |\sigma(t) - s - (\sigma(t) - s)| = 0 \le \varepsilon|\sigma(t) - s|$$
holds for all $s \in \mathbb{T}$.

Exercise 1.14.
(i) Define $f : \mathbb{T} \to \mathbb{R}$ by $f(t) = t^2$ for all $t \in \mathbb{T}$. Find f^Δ.
(ii) Define g by $g(t) = \sqrt{t}$ for all $t \in \mathbb{T}$ with $t > 0$. Find g^Δ.

Exercise 1.15. Using Definition 1.10 show that if $t \in \mathbb{T}^\kappa$ ($t \neq \min \mathbb{T}$) satisfies $\rho(t) = t < \sigma(t)$, then the jump operator σ is not delta differentiable at t.

Some easy and useful relationships concerning the delta derivative are given next.

Theorem 1.16. *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^\kappa$. Then we have the following:*

(i) *If f is differentiable at t, then f is continuous at t.*
(ii) *If f is continuous at t and t is right-scattered, then f is differentiable at t with*
$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$
(iii) *If t is right-dense, then f is differentiable at t iff the limit*
$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number. In this case

$$f^{\Delta}(t) = \lim_{s\to t} \frac{f(t) - f(s)}{t - s}.$$

(iv) *If f is differentiable at t, then*

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t).$$

Proof. Part (i). Assume that f is differentiable at t. Let $\varepsilon \in (0,1)$. Define

$$\varepsilon^* = \varepsilon[1 + |f^{\Delta}(t)| + 2\mu(t)]^{-1}.$$

Then $\varepsilon^* \in (0,1)$. By Definition 1.10 there exists a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - [\sigma(t) - s] f^{\Delta}(t)| \le \varepsilon^* |\sigma(t) - s| \quad \text{for all} \quad s \in U.$$

Therefore we have for all $s \in U \cap (t - \varepsilon^*, t + \varepsilon^*)$

$$\begin{aligned} |f(t) - f(s)| &= \big|\{f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s]\} \\ &\quad -\{f(\sigma(t)) - f(t) - \mu(t) f^{\Delta}(t)\} + (t - s) f^{\Delta}(t)\big| \\ &\le \varepsilon^* |\sigma(t) - s| + \varepsilon^* \mu(t) + |t - s||f^{\Delta}(t)| \\ &\le \varepsilon^* [\mu(t) + |t - s| + \mu(t) + |f^{\Delta}(t)|] \\ &< \varepsilon^* [1 + |f^{\Delta}(t)| + 2\mu(t)] \\ &= \varepsilon. \end{aligned}$$

It follows that f is continuous at t.

Part (ii). Assume f is continuous at t and t is right-scattered. By continuity

$$\lim_{s\to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t} = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

Hence, given $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left| \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} - \frac{f(\sigma(t)) - f(t)}{\mu(t)} \right| \le \varepsilon$$

for all $s \in U$. It follows that

$$\left| [f(\sigma(t)) - f(s)] - \frac{f(\sigma(t)) - f(t)}{\mu(t)} [\sigma(t) - s] \right| \le \varepsilon |\sigma(t) - s|$$

for all $s \in U$. Hence we get the desired result

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

Part (iii). Assume f is differentiable at t and t is right-dense. Let $\varepsilon > 0$ be given. Since f is differentiable at t, there is a neighborhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \le \varepsilon |\sigma(t) - s|$$

for all $s \in U$. Since $\sigma(t) = t$ we have that

$$|[f(t) - f(s)] - f^{\Delta}(t)[t - s]| \le \varepsilon |t - s|$$

for all $s \in U$. It follows that

$$\left| \frac{f(t) - f(s)}{t - s} - f^{\Delta}(t) \right| \le \varepsilon$$

for all $s \in U$, $s \neq t$. Therefore we get the desired result

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

The remaining part of the proof of part (iii) is Exercise 1.17 below.

Part (iv). If $\sigma(t) = t$, then $\mu(t) = 0$ and we have that

$$f(\sigma(t)) = f(t) = f(t) + \mu(t) f^{\Delta}(t).$$

On the other hand if $\sigma(t) > t$, then by (ii)

$$\begin{aligned} f(\sigma(t)) &= f(t) + \mu(t) \cdot \frac{f(\sigma(t)) - f(t)}{\mu(t)} \\ &= f(t) + \mu(t) f^{\Delta}(t), \end{aligned}$$

and the proof of part (iv) is complete. □

Exercise 1.17. Prove the converse part of the statement in part (iii) of Theorem 1.16: Let $t \in \mathbb{T}^{\kappa}$ be right-dense. If

$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$

exists as a finite number, then f is differentiable at t and

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}.$$

Example 1.18. Again we consider the two cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

(i) If $\mathbb{T} = \mathbb{R}$, then Theorem 1.16 (iii) yields that $f : \mathbb{R} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ iff

$$f'(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} \quad \text{exists,}$$

i.e., iff f is differentiable (in the ordinary sense) at t. In this case we then have

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} = f'(t)$$

by Theorem 1.16 (iii).

(ii) If $\mathbb{T} = \mathbb{Z}$, then Theorem 1.16 (ii) yields that $f : \mathbb{Z} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{Z}$ with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(t+1) - f(t)}{1} = f(t+1) - f(t) = \Delta f(t),$$

where Δ is the usual *forward difference operator* defined by the last equation above.

Exercise 1.19. For each of the following functions $f : \mathbb{T} \to \mathbb{R}$, use Theorem 1.16 to find f^{Δ}. Write your final answer in terms of $t \in \mathbb{T}$:

(i) $f(t) = \sigma(t)$ for $t \in \mathbb{T} := \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$;
(ii) $f(t) = t^2$ for $t \in \mathbb{T} := \mathbb{N}_0^{\frac{1}{2}} := \{t = \sqrt{n} : n \in \mathbb{N}_0\}$;
(iii) $f(t) = t^2$ for $t \in \mathbb{T} := \{\frac{n}{2} : n \in \mathbb{N}_0\}$;
(iv) $f(t) = t^3$ for $t \in \mathbb{T} := \mathbb{N}_0^{\frac{1}{3}} := \{t = \sqrt[3]{n} : n \in \mathbb{N}_0\}$.

Next, we would like to be able to find the derivatives of sums, products, and quotients of differentiable functions. This is possible according to the following theorem.

Theorem 1.20. *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^\kappa$. Then:*

(i) *The sum $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*
$$(f+g)^\Delta(t) = f^\Delta(t) + g^\Delta(t).$$
(ii) *For any constant α, $\alpha f : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*
$$(\alpha f)^\Delta(t) = \alpha f^\Delta(t).$$
(iii) *The product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*
$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)).$$
(iv) *If $f(t)f(\sigma(t)) \neq 0$, then $\frac{1}{f}$ is differentiable at t with*
$$\left(\frac{1}{f}\right)^\Delta(t) = -\frac{f^\Delta(t)}{f(t)f(\sigma(t))}.$$
(v) *If $g(t)g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is differentiable at t and*
$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}.$$

Proof. Assume that f and g are delta differentiable at $t \in \mathbb{T}^\kappa$.

Part (i). Let $\varepsilon > 0$. Then there exist neighborhoods U_1 and U_2 of t with
$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \le \frac{\varepsilon}{2}|\sigma(t) - s| \quad \text{for all} \quad s \in U_1$$
and
$$|g(\sigma(t)) - g(s) - g^\Delta(t)(\sigma(t) - s)| \le \frac{\varepsilon}{2}|\sigma(t) - s| \quad \text{for all} \quad s \in U_2.$$
Let $U = U_1 \cap U_2$. Then we have for all $s \in U$
$$\begin{aligned} &|(f+g)(\sigma(t)) - (f+g)(s) - [f^\Delta(t) + g^\Delta(t)](\sigma(t) - s)| \\ &= |f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s) + g(\sigma(t)) - g(s) - g^\Delta(t)(\sigma(t) - s)| \\ &\le |f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| + |g(\sigma(t)) - g(s) - g^\Delta(t)(\sigma(t) - s)| \\ &\le \frac{\varepsilon}{2}|\sigma(t) - s| + \frac{\varepsilon}{2}|\sigma(t) - s| \\ &= \varepsilon|\sigma(t) - s|. \end{aligned}$$
Therefore $f + g$ is differentiable at t and $(f+g)^\Delta = f^\Delta + g^\Delta$ holds at t.

Part (iii). Let $\varepsilon \in (0, 1)$. Define $\varepsilon^* = \varepsilon[1 + |f(t)| + |g(\sigma(t))| + |g^\Delta(t)|]^{-1}$. Then $\varepsilon^* \in (0, 1)$ and hence there exist neighborhoods U_1, U_2, and U_3 of t such that
$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \le \varepsilon^*|\sigma(t) - s| \quad \text{for all} \quad s \in U_1,$$
$$|g(\sigma(t)) - g(s) - g^\Delta(t)(\sigma(t) - s)| \le \varepsilon^*|\sigma(t) - s| \quad \text{for all} \quad s \in U_2,$$
and (from Theorem 1.16 (i))
$$|f(t) - f(s)| \le \varepsilon^* \quad \text{for all} \quad s \in U_3.$$

Put $U = U_1 \cap U_2 \cap U_3$ and let $s \in U$. Then

$$\begin{aligned}
&\left|(fg)(\sigma(t)) - (fg)(s) - [f^\Delta(t)g(\sigma(t)) + f(t)g^\Delta(t)](\sigma(t) - s)\right| \\
= \;& \big|[f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)]g(\sigma(t)) \\
& +[g(\sigma(t)) - g(s) - g^\Delta(t)(\sigma(t) - s)]f(t) \\
& +[g(\sigma(t)) - g(s) - g^\Delta(t)(\sigma(t) - s)][f(s) - f(t)] \\
& +(\sigma(t) - s)g^\Delta(t)[f(s) - f(t)]\big| \\
\leq \;& \varepsilon^*|\sigma(t) - s||g(\sigma(t))| + \varepsilon^*|\sigma(t) - s||f(t)| \\
& +\varepsilon^*\varepsilon^*|\sigma(t) - s| + \varepsilon^*|\sigma(t) - s||g^\Delta(t)| \\
= \;& \varepsilon^*|\sigma(t) - s|[|g(\sigma(t))| + |f(t)| + \varepsilon^* + |g^\Delta(t)|] \\
\leq \;& \varepsilon^*|\sigma(t) - s|[1 + |f(t)| + |g(\sigma(t))| + |g^\Delta(t)|] \\
= \;& \varepsilon|\sigma(t) - s|.
\end{aligned}$$

Thus $(fg)^\Delta = f^\Delta g^\sigma + fg^\Delta$ holds at t. The other product rule in part (iii) of this theorem follows from this last equation by interchanging the functions f and g.

For the quotient formula (v), we use (ii) and (iv) to calculate

$$\begin{aligned}
\left(\frac{f}{g}\right)^\Delta(t) &= \left(f \cdot \frac{1}{g}\right)^\Delta(t) \\
&= f(t)\left(\frac{1}{g}\right)^\Delta(t) + f^\Delta(t)\frac{1}{g(\sigma(t))} \\
&= -f(t)\frac{g^\Delta(t)}{g(t)g(\sigma(t))} + f^\Delta(t)\frac{1}{g(\sigma(t))} \\
&= \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}.
\end{aligned}$$

The reader is asked in Exercise 1.21 to prove (ii) and (iv). □

Exercise 1.21. Prove parts (ii) and (iv) of Theorem 1.20.

Exercise 1.22. Prove that if x, y, and z are delta differentiable at t, then

$$(xyz)^\Delta = x^\Delta yz + x^\sigma y^\Delta z + x^\sigma y^\sigma z^\Delta$$

holds at t. Write down the generalization of this formula for n functions.

Exercise 1.23. We have by Theorem 1.20 (iii)

$$(f^2)^\Delta = (f \cdot f)^\Delta = f^\Delta f + f^\sigma f^\Delta = (f + f^\sigma)f^\Delta. \tag{1.1}$$

Give the generalization of this formula for the derivative of the $(n+1)$st power of f, $n \in \mathbb{N}$, i.e., for $(f^{n+1})^\Delta$.

Theorem 1.24. *Let α be constant and $m \in \mathbb{N}$.*

(i) *For f defined by $f(t) = (t - \alpha)^m$ we have*

$$f^\Delta(t) = \sum_{\nu=0}^{m-1} (\sigma(t) - \alpha)^\nu (t - \alpha)^{m-1-\nu}.$$

(ii) *For g defined by $g(t) = \frac{1}{(t-\alpha)^m}$ we have*

$$g^{\Delta}(t) = -\sum_{\nu=0}^{m-1} \frac{1}{(\sigma(t)-\alpha)^{m-\nu}(t-\alpha)^{\nu+1}},$$

provided $(t-\alpha)(\sigma(t)-\alpha) \neq 0$.

Proof. We will prove the first formula by induction. If $m = 1$, then $f(t) = t - \alpha$, and clearly $f^{\Delta}(t) = 1$ holds by Example 1.13 (i), (ii), and Theorem 1.20 (i). Now we assume that

$$f^{\Delta}(t) = \sum_{\nu=0}^{m-1} (\sigma(t)-\alpha)^{\nu}(t-\alpha)^{m-1-\nu}$$

holds for $f(t) = (t-\alpha)^m$ and let $F(t) = (t-\alpha)^{m+1} = (t-\alpha)f(t)$. We use the product rule, Theorem 1.20 (iii), to obtain

$$\begin{aligned} F^{\Delta}(t) &= f(\sigma(t)) + (t-\alpha)f^{\Delta}(t) \\ &= (\sigma(t)-\alpha)^m + (t-\alpha)\sum_{\nu=0}^{m-1}(\sigma(t)-\alpha)^{\nu}(t-\alpha)^{m-1-\nu} \\ &= (\sigma(t)-\alpha)^m + \sum_{\nu=0}^{m-1}(\sigma(t)-\alpha)^{\nu}(t-\alpha)^{m-\nu} \\ &= \sum_{\nu=0}^{m}(\sigma(t)-\alpha)^{\nu}(t-\alpha)^{m-\nu}. \end{aligned}$$

Hence, by mathematical induction, part (i) holds.

Next, for $g(t) = \frac{1}{(t-\alpha)^m} = \frac{1}{f(t)}$ we apply Theorem 1.20 (iv) to obtain

$$\begin{aligned} g^{\Delta}(t) &= -\frac{f^{\Delta}(t)}{f(t)f^{\sigma}(t)} \\ &= -\frac{\sum_{\nu=0}^{m-1}(\sigma(t)-\alpha)^{\nu}(t-\alpha)^{m-1-\nu}}{(t-\alpha)^m(\sigma(t)-\alpha)^m} \\ &= -\sum_{\nu=0}^{m-1}\frac{1}{(t-\alpha)^{\nu+1}(\sigma(t)-\alpha)^{m-\nu}}, \end{aligned}$$

provided $(t-\alpha)(\sigma(t)-\alpha) \neq 0$. □

Example 1.25. The derivative of t^2 is

$$t + \sigma(t).$$

The derivative of $1/t$ is

$$-\frac{1}{t\sigma(t)}.$$

Exercise 1.26. Use Theorem 1.24 to find the derivatives in Exercise 1.19.

We define higher order derivatives of functions on time scales in the usual way.

Definition 1.27. For a function $f : \mathbb{T} \to \mathbb{R}$ we shall talk about the second derivative $f^{\Delta\Delta}$ provided f^{Δ} is differentiable on $\mathbb{T}^{\kappa^2} = (\mathbb{T}^{\kappa})^{\kappa}$ with derivative $f^{\Delta\Delta} = (f^{\Delta})^{\Delta} : \mathbb{T}^{\kappa^2} \to \mathbb{R}$. Similarly we define higher order derivatives $f^{\Delta^n} : \mathbb{T}^{\kappa^n} \to \mathbb{R}$. Finally, for $t \in \mathbb{T}$, we denote $\sigma^2(t) = \sigma(\sigma(t))$ and $\rho^2(t) = \rho(\rho(t))$, and $\sigma^n(t)$ and $\rho^n(t)$ for $n \in \mathbb{N}$ are defined accordingly. For convenience we also put

$$\sigma^0(t) = t,\ \rho^0(t) = t,\ f^{\Delta^0} = f, \quad \text{and} \quad \mathbb{T}^{\kappa^0} = \mathbb{T}.$$

Exercise 1.28. Find the second derivative of each of the functions given in Exercise 1.19.

Exercise 1.29. Find the second derivative of f on an arbitrary time scale:

(i) $f(t) \equiv 1$;
(ii) $f(t) = t$;
(iii) $f(t) = t^2$.

Exercise 1.30. For an arbitrary time scale, try to find a function f such that $f^{\Delta\Delta} = 1$.

Example 1.31. In general, fg is not twice differentiable even if both f and g are twice differentiable. We have

$$(fg)^{\Delta} = f^{\Delta}g + f^{\sigma}g^{\Delta}.$$

If f and g are twice differentiable, and if f^{σ} is differentiable, then

$$\begin{aligned} (fg)^{\Delta} &= (f^{\Delta}g + f^{\sigma}g^{\Delta})^{\Delta} \\ &= f^{\Delta\Delta}g + f^{\Delta^{\sigma}}g^{\Delta} + f^{\sigma^{\Delta}}g^{\Delta} + f^{\sigma^{\sigma}}g^{\Delta\Delta} \\ &= f^{\Delta\Delta}g + (f^{\Delta\sigma} + f^{\sigma\Delta})g^{\Delta} + f^{\sigma\sigma}g^{\Delta\Delta}, \end{aligned}$$

where we wrote $f^{\Delta\sigma}$ for $f^{\Delta^{\sigma}}$ etc.; we shall also use such notation in the sequel for combinations of more than two "exponents" of the form Δ or σ. The formula for the nth derivative under certain conditions is given in the following result.

Theorem 1.32 (Leibniz Formula). *Let $S_k^{(n)}$ be the set consisting of all possible strings of length n, containing exactly k times σ and $n-k$ times Δ. If*

$$f^{\Lambda} \quad \textit{exists for all} \quad \Lambda \in S_k^{(n)},$$

then

$$(fg)^{\Delta^n} = \sum_{k=0}^{n} \left[\sum_{\Lambda \in S_k^{(n)}} f^{\Lambda} \right] g^{\Delta^k} \tag{1.2}$$

holds for all $n \in \mathbb{N}$.

Proof. We will show (1.2) by induction. First, if $n = 1$, then (1.2) is true by the product rule from Theorem 1.20 (iii). (With the convention that $\sum_{\Lambda\in\emptyset} f^{\Lambda} = f$, (1.2) also holds for $n = 0$.) Next, we assume that (1.2) is true for $n = m \in \mathbb{N}$.

Then, using Theorem 1.20 (i) and (iii), we obtain

$$\begin{aligned}
(fg)^{\Delta^{m+1}} &= \left\{\sum_{k=0}^{m}\left[\sum_{\Lambda\in S_k^{(m)}} f^{\Lambda}\right] g^{\Delta^k}\right\}^{\Delta} \\
&= \sum_{k=0}^{m}\left\{\left[\sum_{\Lambda\in S_k^{(m)}} f^{\Lambda}\right]^{\sigma} g^{\Delta^{k+1}} + \left[\sum_{\Lambda\in S_k^{(m)}} f^{\Lambda}\right]^{\Delta} g^{\Delta^k}\right\} \\
&= \sum_{k=1}^{m+1}\left[\sum_{\Lambda\in S_{k-1}^{(m)}} f^{\Lambda\sigma}\right] g^{\Delta^k} + \sum_{k=0}^{m}\left[\sum_{\Lambda\in S_k^{(m)}} f^{\Lambda\Delta}\right] g^{\Delta^k} \\
&= \left[\sum_{\Lambda\in S_m^{(m)}} f^{\Lambda\sigma}\right] g^{\Delta^{m+1}} + \left[\sum_{\Lambda\in S_0^{(m)}} f^{\Lambda\Delta}\right] g \\
&\quad + \sum_{k=1}^{m}\left[\sum_{\Lambda\in S_{k-1}^{(m)}} f^{\Lambda\sigma} + \sum_{\Lambda\in S_k^{(m)}} f^{\Lambda\Delta}\right] g^{\Delta^k} \\
&= \left[\sum_{\Lambda\in S_{m+1}^{(m+1)}} f^{\Lambda}\right] g^{\Delta^{m+1}} + \left[\sum_{\Lambda\in S_0^{(m+1)}} f^{\Lambda}\right] g + \sum_{k=1}^{m}\left[\sum_{\Lambda\in S_k^{(m+1)}} f^{\Lambda}\right] g^{\Delta^k} \\
&= \sum_{k=0}^{m+1}\left[\sum_{\Lambda\in S_k^{(m+1)}} f^{\Lambda}\right] g^{\Delta^k}
\end{aligned}$$

so that (1.2) holds for $n = m+1$. By the principle of mathematical induction, (1.2) holds for all $n \in \mathbb{N}_0$. ☐

Example 1.33. If $\mathbb{T} = \mathbb{R}$, then

$$f^{\Lambda} = f^{(n-k)} \quad \text{for all} \quad \Lambda \in S_k^{(n)},$$

where $f^{(n)}$ denotes the nth (usual) derivative of f, if it exists, and since

$$\left|S_k^{(n)}\right| = \binom{n}{k},$$

where $|M|$ denotes the cardinality of the set M, we have

$$\sum_{\Lambda\in S_k^{(n)}} f^{\Lambda} = \sum_{\Lambda\in S_k^{(n-k)}} f^{(n)} = f^{(n-k)} \sum_{\Lambda\in S_k^{(n)}} 1 = \binom{n}{k} f^{(n-k)}$$

and therefore

$$(fg)^{\Delta^n} = \sum_{k=0}^{n}\left[\sum_{\Lambda\in S_k^{(n)}} f^{\Lambda}\right] g^{\Delta^k} = \sum_{k=0}^{n}\binom{n}{k} f^{(n-k)} g^{(k)}.$$

This is the usual Leibniz formula from calculus.

Exercise 1.34. Use Theorem 1.32 to find $\Delta^n(fg)$, i.e., $(fg)^{\Delta^n}$ if $\mathbb{T} = \mathbb{Z}$.

Exercise 1.35. Show that in general, even if both f^{Δ^σ} and f^{σ^Δ} exist,

$$f^{\Delta^\sigma} = f^{\sigma^\Delta} \tag{1.3}$$

does not hold. Is (1.3) true for $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$? Give a sufficient condition that guarantees that (1.3) holds.

Exercise 1.36. Suppose μ is differentiable.

(i) If f^{Δ^σ} and f^{σ^Δ} both exist, give a formula that actually relates these two functions.
(ii) Give a similar formula that relates the functions (if they exist) $f^{\sigma\sigma\Delta}$, $f^{\sigma\Delta\sigma}$, and $f^{\Delta\sigma\sigma}$.
(iii) Give the corresponding formula that relates the functions (if they exist) $f^{\sigma^n\Delta}$ and $f^{\Delta\sigma^n}$.

1.3. Examples and Applications

In this section we will discuss some examples of time scales that are considered throughout this book.

Example 1.37. Let $h > 0$ and

$$\mathbb{T} = h\mathbb{Z} = \{hk : \ k \in \mathbb{Z}\}.$$

Then we have for $t \in \mathbb{T}$

$$\sigma(t) = \inf\{s \in \mathbb{T} : \ s > t\} = \inf\{t + nh : \ n \in \mathbb{N}\} = t + h$$

and similarly $\rho(t) = t - h$. Hence every point $t \in \mathbb{T}$ is isolated and

$$\mu(t) = \sigma(t) - t = t + h - t \equiv h \quad \text{for all} \quad t \in \mathbb{T}$$

so that μ in this example is constant. For a function $f : \mathbb{T} \to \mathbb{R}$ we have

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(t+h) - f(t)}{h} \quad \text{for all} \quad t \in \mathbb{T}.$$

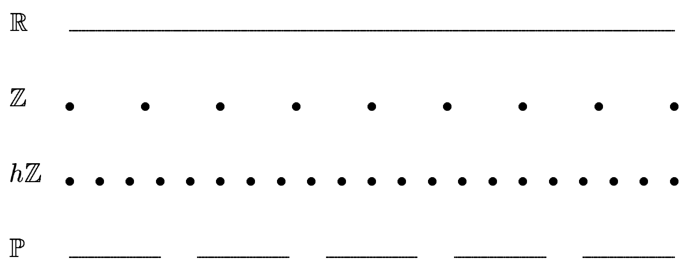
Next,

$$\begin{aligned} f^{\Delta\Delta}(t) &= \frac{f^\Delta(\sigma(t)) - f^\Delta(t)}{\mu(t)} \\ &= \frac{f^\Delta(t+h) - f^\Delta(t)}{h} \\ &= \frac{\frac{f(t+2h)-f(t+h)}{h} - \frac{f(t+h)-f(t)}{h}}{h} \\ &= \frac{f(t+2h) - f(t+h) - f(t+h) + f(t)}{h^2} \\ &= \frac{f(t+2h) - 2f(t+h) + f(t)}{h^2}. \end{aligned}$$

It would be too tedious to calculate $f^{\Delta^n}(t)$ in a similar way, so we consider another method. First, note that

$$\sigma^n(t) = t + nh \quad \text{and} \quad \rho^n(t) = t - nh \quad \text{for all} \quad n \in \mathbb{N}_0.$$

Figure 1.2. Some Time Scales



We now introduce an operator Δ_h by

$$\Delta_h = \frac{1}{h}(\sigma - I), \quad I \text{ being the identity operator.}$$

Recall that the binomial theorem says that

$$\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = (a+b)^n.$$

We use an operator version of this binomial theorem to obtain the nth power of Δ_h as

$$\Delta_h^n = \frac{1}{h^n}(\sigma - I)^n = \frac{1}{h^n} \sum_{k=0}^{n} \binom{n}{k} \sigma^k (-1)^{n-k}.$$

Applying the obtained operator to the function f, we find

$$f^{\Delta^n}(t) = \frac{1}{h^n} \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(t+kh).$$

This time scale is of particular interest: It is in some cases possible to obtain "continuous" results, i.e., results for $\mathbb{T} = \mathbb{R}$, via letting h tend to zero from above in the corresponding "discrete" results, i.e., results for $\mathbb{T} = h\mathbb{Z}$.

We now give some examples of time scales with nonconstant graininess.

Example 1.38. Let $a, b > 0$ and consider the time scale

$$\mathbb{P}_{a,b} = \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a].$$

Then

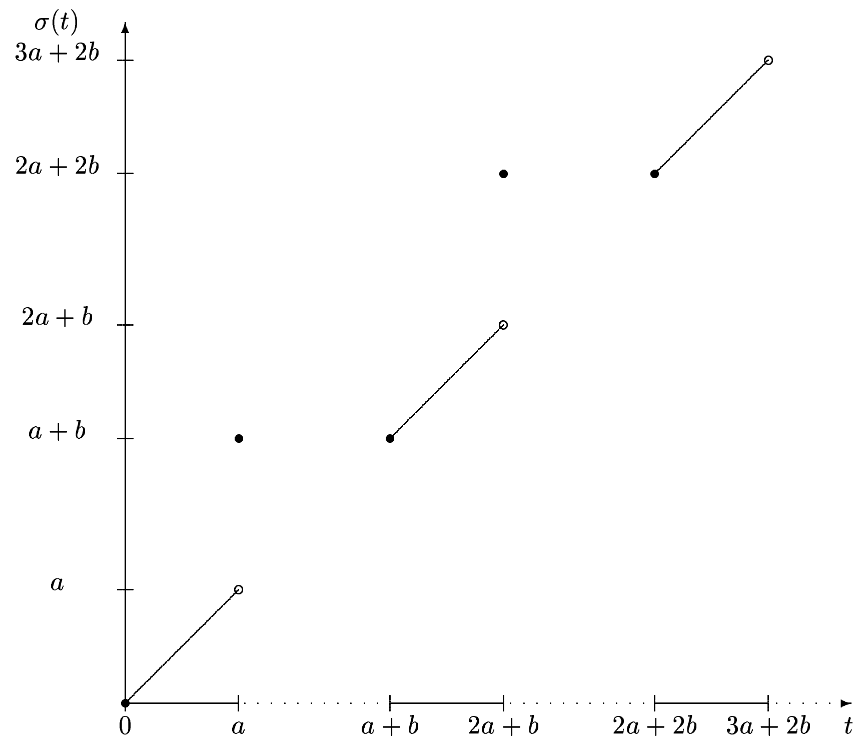
$$\sigma(t) = \begin{cases} t & \text{if} \quad t \in \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a) \\ t+b & \text{if} \quad t \in \bigcup_{k=0}^{\infty} \{k(a+b)+a\} \end{cases}$$

and

$$\mu(t) = \begin{cases} 0 & \text{if} \quad t \in \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a) \\ b & \text{if} \quad t \in \bigcup_{k=0}^{\infty} \{k(a+b)+a\}. \end{cases}$$

See Figure 1.3 for a graph of the forward jump operator.

Figure 1.3. Forward jump operator for $\mathbb{P}_{a,b}$



Example 1.39. Assume that the life span of a certain species is one unit of time. Suppose that just before the species dies out, eggs are laid which are hatched one unit of time later. Hence we are interested in the time scale

$$\mathbb{P}_{1,1} = \bigcup_{k=0}^{\infty} [2k, 2k+1].$$

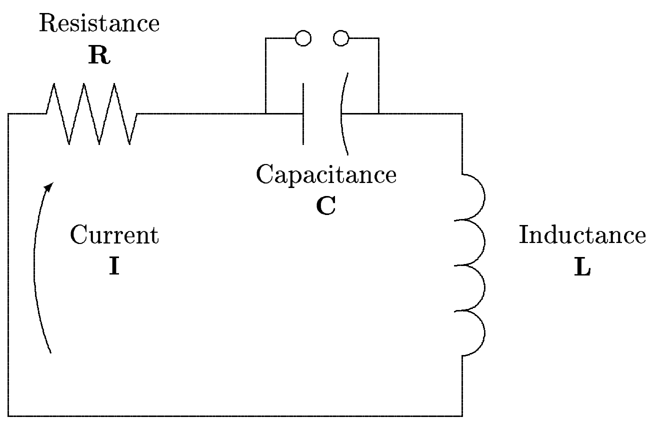
For this time scale,

$$\mu(t) = \begin{cases} 0 & \text{for} \quad t \in \bigcup_{k=0}^{\infty}[2k, 2k+1) \\ 1 & \text{for} \quad t \in \bigcup_{k=0}^{\infty}\{2k+1\}. \end{cases}$$

For a specific example of this type see Christiansen and Fenchel [**96**, page 7ff]. A couple of examples of this type, where the time scale consists of a sequence of disjoint closed intervals, are the 17 year cicada *magicicada septendecim* which lives as a larva for 17 years and as an adult for perhaps a week, and the common mayfly *stenonema canadense* which lives as a larva for a year and as an adult for less than a day.

Example 1.40 (S. Keller [**190**, Beispiel 2.1.13])**.** Consider a simple electric circuit with resistance R, inductance L, and capacitance C (see Figure 1.4). Suppose we discharge the capacitor periodically every time unit and assume that the discharging

Figure 1.4. An Electric Circuit



takes $\delta > 0$ (but small) time units. Then this simulation can be modeled using the time scale

$$\mathbb{P}_{1-\delta,\delta} = \bigcup_{k\in\mathbb{N}_0} [k, k+1-\delta].$$

If $Q(t)$ is the total charge on the capacitor at time t and $I(t)$ is the current as a function of time t, then we have

$$Q^\Delta(t) = \begin{cases} bQ(t) & \text{if} \quad t \in \bigcup_{k\in\mathbb{N}} \{k-\delta\} \\ I & \text{otherwise} \end{cases}$$

and

$$I^\Delta(t) = \begin{cases} 0 & \text{if} \quad t \in \bigcup_{k\in\mathbb{N}} \{k-\delta\} \\ -\frac{1}{LC}Q(t) - \frac{R}{L}I(t) & \text{otherwise,} \end{cases}$$

where b is a constant satisfying $-1 < b\delta < 0$.

Example 1.41. Let $q > 1$ and

$$q^{\mathbb{Z}} := \{q^k : k \in \mathbb{Z}\} \quad \text{and} \quad \overline{q^{\mathbb{Z}}} := q^{\mathbb{Z}} \cup \{0\}.$$

Here we consider the time scale $\mathbb{T} = \overline{q^{\mathbb{Z}}}$. We have

$$\sigma(t) = \inf\{q^n : n \in [m+1, \infty)\} = q^{m+1} = qq^m = qt$$

if $t = q^m \in \mathbb{T}$ and obviously $\sigma(0) = 0$. So we obtain

$$\sigma(t) = qt \quad \text{and} \quad \rho(t) = \frac{t}{q} \quad \text{for all} \quad t \in \mathbb{T}$$

and consequently

$$\mu(t) = \sigma(t) - t = (q-1)t \quad \text{for all} \quad t \in \mathbb{T}.$$

Hence 0 is a right-dense minimum and every other point in $\mathbb{T}$ is isolated. For a function $f : \mathbb{T} \to \mathbb{R}$ we have

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = \frac{f(qt) - f(t)}{(q-1)t} \quad \text{for all} \quad t \in \mathbb{T} \setminus \{0\}$$

and

$$f^{\Delta}(0) = \lim_{s \to 0} \frac{f(0) - f(s)}{0 - s} = \lim_{s \to 0} \frac{f(s) - f(0)}{s}$$

provided this limit exists. Now we calculate the second derivative of f at $t \neq 0$ (refer to Definition 1.27 for how $f^{\Delta\Delta}$ is defined) as

$$\begin{aligned} f^{\Delta\Delta}(t) &= \frac{f^{\Delta}(\sigma(t)) - f^{\Delta}(t)}{\mu(t)} \\ &= \frac{f^{\Delta}(qt) - f^{\Delta}(t)}{(q-1)t} \\ &= \frac{\frac{f(q^2t) - f(qt)}{q(q-1)t} - \frac{f(qt) - f(t)}{(q-1)t}}{(q-1)t} \\ &= \frac{f(q^2t) - f(qt) - qf(qt) + qf(t)}{q(q-1)^2 t} \\ &= \frac{f(q^2t) - (q+1)f(qt) + qf(t)}{q(q-1)^2 t}. \end{aligned}$$

Notice that $\mu(t) = t$ above in the particular case $q = 2$.

Exercise 1.42. Let $q > 1$. For the time scale $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, evaluate

(i) σ^{Δ};
(ii) μ^{Δ}.

Exercise 1.43. Find f^{Δ^3} for the time scale $\mathbb{T} = \overline{q^{\mathbb{Z}}}$. Find f^{Δ^4} and finally find a formula for f^{Δ^n} for any natural number n.

Example 1.44. Consider the time scale

$$\mathbb{T} = \mathbb{N}_0^2 = \{n^2 : n \in \mathbb{N}_0\}.$$

We have $\sigma(n^2) = (n+1)^2$ for $n \in \mathbb{N}_0$ and

$$\mu(n^2) = \sigma(n^2) - n^2 = (n+1)^2 - n^2 = 2n + 1.$$

Hence

$$\sigma(t) = (\sqrt{t} + 1)^2 \quad \text{and} \quad \mu(t) = 1 + 2\sqrt{t} \quad \text{for} \quad t \in \mathbb{T}.$$

Example 1.45. Let H_n be the so-called *harmonic numbers*

$$H_0 = 0 \quad \text{and} \quad H_n = \sum_{k=1}^{n} \frac{1}{k} \quad \text{for } n \in \mathbb{N}.$$

Consider the time scale

$$\mathbb{T} = \{H_n : n \in \mathbb{N}_0\}.$$

We have $\sigma(H_n) = H_{n+1}$ for all $n \in \mathbb{N}_0$, $\rho(H_n) = H_{n-1}$ when $n \in \mathbb{N}$, and $\rho(H_0) = H_0$. The graininess is given by

$$\mu(H_n) = \sigma(H_n) - H_n = H_{n+1} - H_n = \frac{1}{n+1}$$

Table 1.2. Examples of Time Scales

$\mathbb{T}$	$\mu(t)$	$\sigma(t)$	$\rho(t)$
$\mathbb{R}$	0	t	t
$\mathbb{Z}$	1	$t+1$	$t-1$
$h\mathbb{Z}$	h	$t+h$	$t-h$
$q^{\mathbb{N}}$	$(q-1)t$	qt	$\frac{t}{q}$
$2^{\mathbb{N}}$	t	$2t$	$\frac{t}{2}$
$\mathbb{N}_0^2$	$2\sqrt{t}+1$	$(\sqrt{t}+1)^2$	$(\sqrt{t}-1)^2$

for all $n \in \mathbb{N}_0$. If $f : \mathbb{T} \to \mathbb{R}$ is a function, then

$$f^{\Delta}(H_n) = \frac{f(H_{n+1}) - f(H_n)}{\mu(H_n)} = (n+1)\Delta f(H_n).$$

Example 1.46. We let $\{\alpha_n\}_{n\in\mathbb{N}_0}$ be a sequence of real numbers with $\alpha_n > 0$ for all $n \in \mathbb{N}$ and put

$$t_n = \sum_{k=0}^{n-1} \alpha_k.$$

Consider the time scale

$$\mathbb{T} = \{t_n : n \in \mathbb{N}\}$$

if $\sum_{k=0}^{\infty} \alpha_k = \infty$ or

$$\mathbb{T} = \{t_n : n \in \mathbb{N}\} \cup \{L\}$$

if $\sum_{k=0}^{\infty} \alpha_k = L$ converges. We have

$$\sigma(t_n) = t_{n+1} \quad \text{and} \quad \mu(t_n) = \alpha_n$$

for all $n \in \mathbb{N}$. For a function $y : \mathbb{T} \to \mathbb{R}$ we find

$$y^{\Delta}(t_n) = \frac{y(t_{n+1}) - y(t_n)}{\alpha_n} = \frac{\Delta y(t_n)}{\alpha_n}$$

for all $n \in \mathbb{N}$.

We remark that using the harmonic series above corresponds to Example 1.45, while using the geometric series corresponds to Example 1.41.

Example 1.47 (The Cantor Set)**.** Consider $K_0 = [0,1]$. We obtain a subset K_1 of K_0 by removing the open "middle third" of K_0, i.e., the open interval $(1/3, 2/3)$, from K_0. K_2 is obtained by removing the two open middle thirds of K_1, i.e., the two open intervals $(1/9, 2/9)$ and $(7/9, 8/9)$ from K_1. Proceeding in this manner, we obtain a sequence $\{K_n\}_{n\in\mathbb{N}_0}$ of subsets of $[0,1]$. See Figure 1.5 for K_0, K_1, K_2, and K_3. The Cantor set C is now defined as

$$C = \bigcap_{n=0}^{\infty} K_n$$

and hence is closed. Therefore $\mathbb{T} = C$ is a time scale. Each $x \in [0,1]$ can be represented in its *ternary* expansion as

$$x = \sum_{k=1}^{\infty} \frac{a_k}{3^k}, \qquad \text{where} \qquad a_k \in \{0,1,2\} \text{ for each } k \in \mathbb{N}.$$

It is known that a number x is an element of C if and only if it can be represented by a ternary expansion, where the a_k are either 0 or 2 (see e.g., [**142**, page 38]). Let L denote the set of all the left-hand end points of the open intervals removed, i.e.,

$$L = \left\{ \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}} : \ m \in \mathbb{N} \text{ and } a_k \in \{0,2\} \text{ for all } 1 \le k \le m \right\}.$$

Then $L \subset \mathbb{T}$. The set of all right-hand end points of the open intervals removed is given by

$$R = \left\{ \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{2}{3^{m+1}} : \ m \in \mathbb{N} \text{ and } a_k \in \{0,2\} \text{ for all } 1 \le k \le m \right\},$$

and we also have $R \subset \mathbb{T}$. It follows that

$$\sigma(t) = t + \frac{1}{3^{m+1}} \qquad \text{whenever} \qquad t = \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}} \in L.$$

Each point $t \in \mathbb{T} \setminus L$ has other points of $\mathbb{T}$ in any neighborhood of t, and therefore satisfies $\sigma(t) = t$. Altogether,

$$\sigma(t) = \begin{cases} t + \frac{1}{3^{m+1}} & \text{if} \quad t = \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}} \in L \\ t & \text{if} \quad t \in \mathbb{T} \setminus L \end{cases}$$

and similarly

$$\rho(t) = \begin{cases} t - \frac{1}{3^{m+1}} & \text{if} \quad t = \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{2}{3^{m+1}} \in R \\ t & \text{if} \quad t \in \mathbb{T} \setminus R. \end{cases}$$

Now we obtain the graininess function μ of the Cantor set as

$$\mu(t) = \begin{cases} \frac{1}{3^{m+1}} & \text{if} \quad t = \sum_{k=1}^{m} \frac{a_k}{3^k} + \frac{1}{3^{m+1}} \in L \\ 0 & \text{if} \quad t \in \mathbb{T} \setminus L. \end{cases}$$

Hence L consists of the right-scattered elements of $\mathbb{T}$, and R consists of the left-scattered elements of $\mathbb{T}$. Thus, $\mathbb{T}$ does not contain any isolated points.

We now discuss some results for the time scale $\mathbb{T} = \mathbb{Z}$.

Definition 1.48. Let $t \in \mathbb{C}$ (i.e., t is a complex number) and $k \in \mathbb{Z}$. The *factorial function* $t^{(k)}$ is defined as follows:

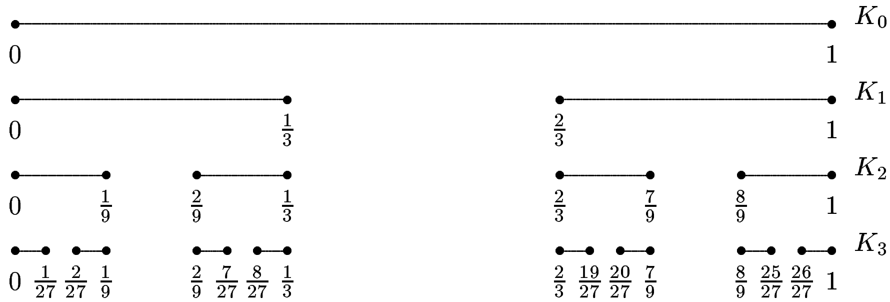
(i) If $k \in \mathbb{N}$, then

$$t^{(k)} = t(t-1)\cdots(t-k+1).$$

(ii) If $k = 0$, then

$$t^{(0)} = 1.$$

Figure 1.5. The Cantor Set



(iii) If $-k \in \mathbb{N}$, then

$$t^{(k)} = \frac{1}{(t+1)(t+2)\cdots(t-k)}$$

for $t \neq -1, -2, \cdots, k$.

In general

$$t^{(k)} := \frac{\Gamma(t+1)}{\Gamma(t-k+1)} \tag{1.4}$$

for all $t, k \in \mathbb{C}$ such that the right-hand side of (1.4) makes sense, where Γ is the gamma function. See [**191**] for some results concerning the gamma and factorial functions.

Exercise 1.49. Show that the general definition (1.4) of the factorial function $t^{(k)}$ gives parts (i), (ii), and (iii) in Definition 1.48 as special cases.

Exercise 1.50. Show that for any constant c

$$u(t) = ca^t \frac{\Gamma(t-t_1)\Gamma(t-t_2)\cdots\Gamma(t-t_n)}{\Gamma(t-s_1)\Gamma(t-s_2)\cdots\Gamma(t-s_m)}$$

is a solution of the recurrence relation

$$u(t+1) = a\frac{(t-t_1)(t-t_2)\cdots(t-t_n)}{(t-s_1)(t-s_2)\cdots(t-s_m)}u(t),$$

where $a, t_1, \ldots, t_n, s_1, \ldots, s_m$ are real constants and $n, m \in \mathbb{N}$. Use this to solve the difference equations

(i) $\Delta u = \frac{4t+6}{t^2+5t+6}u$, where $t \in \mathbb{N}$;
(ii) $\Delta u = \frac{2t+5}{2t+1}u$, where $t \in \mathbb{N}_0$.

Next we define a general binomial coefficient $\binom{\alpha}{\beta}$.

Definition 1.51. We define the *binomial coefficient* $\binom{\alpha}{\beta}$ by

$$\binom{\alpha}{\beta} = \frac{\alpha^{(\beta)}}{\Gamma(\beta+1)}$$

for all $\alpha, \beta \in \mathbb{C}$ such that the right-hand side of this equation makes sense.

Exercise 1.52. Assume $\alpha, k \in \mathbb{C}$ and $^\Delta$ is differentiation with respect to t on the time scale $\mathbb{T} = \mathbb{Z}$. Show that

(i) $\left[(t+\alpha)^{(k)}\right]^\Delta = k(t+\alpha)^{(k-1)}$;
(ii) $(\alpha^t)^\Delta = (\alpha - 1)\alpha^t$;
(iii) $\binom{t}{\alpha}^\Delta = \binom{t}{\alpha-1}$.

Exercise 1.53. Prove the following well-known formula concerning binomial coefficients

$$\binom{\alpha}{\beta} + \binom{\alpha}{\beta+1} = \binom{\alpha+1}{\beta+1}.$$

Exercise 1.54. Introduce some of the above concepts for the time scale $\mathbb{T} = h\mathbb{Z}$ and prove some of the above results for this time scale.

We conclude this section by giving some examples concerning the jump operator.

Example 1.55 (σ is in general not continuous)**.** Here we present an example of a time scale $\mathbb{T}$ whose jump function $\sigma : \mathbb{T} \to \mathbb{T}$ is not continuous at a point $t \in \mathbb{T}$ which is left-dense and right-scattered at the same time. This example is due to Douglas Anderson. Let

$$\mathbb{T} = \{t_n : \; n \in \mathbb{N}\} \cup \mathbb{N}_0, \quad t_n = -\frac{1}{n}.$$

Then

$$\sigma(t_n) = t_{n+1} = -\frac{1}{n+1} \to 0 \neq 1 = \sigma(0), \quad n \to \infty,$$

and hence $\lim_{s \to 0} \sigma(s) \neq \sigma(0)$ so σ is not continuous at 0. According to Theorem 1.16 (i), σ is not differentiable at 0 either, and this can also be shown directly in this case using the definition of differentiability, Definition 1.10. However, note that σ is continuous at right-dense points and that $\lim_{s \to t^-} \sigma(s)$ exists at left-dense points $t \in \mathbb{T}$.

Example 1.56 (σ is in general not differentiable)**.** Here we present an example of a time scale $\mathbb{T}$ whose jump function $\sigma : \mathbb{T} \to \mathbb{T}$ is continuous but not differentiable at a right-dense point $t \in \mathbb{T}$. Let

$$\mathbb{T} = \{t_n : \; n \in \mathbb{N}_0\} \cup \{0, -1\}, \quad t_n = \left(\frac{1}{2}\right)^{2^n}.$$

Then

$$\sigma(t_n) = t_{n-1} \to 0 = \sigma(0), \quad n \to \infty,$$

and hence $\lim_{s \to 0} \sigma(s) = \sigma(0)$ so σ is continuous at 0. But

$$\begin{aligned} \lim_{s \to 0} \frac{\sigma(\sigma(0)) - \sigma(s)}{\sigma(0) - s} &= \lim_{s \to 0} \frac{\sigma(s)}{s} \\ &= \lim_{s \to 0} \frac{\sqrt{s}}{s} \\ &= \lim_{s \to 0} \frac{1}{\sqrt{s}} = \infty \end{aligned}$$

so that σ is not differentiable at 0 (regardless, in fact, whether 0 is left-dense or left-scattered). Note that t is twice differentiable at 0 (see Example 1.13) while $t \cdot t = t^2$ is not.

1.4. Integration

In order to describe classes of functions that are "integrable", we introduce the following two concepts.

Definition 1.57. A function $f : \mathbb{T} \to \mathbb{R}$ is called *regulated* provided its right-sided limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$.

Definition 1.58. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted in this book by

$$\mathrm{C_{rd}} = \mathrm{C_{rd}}(\mathbb{T}) = \mathrm{C_{rd}}(\mathbb{T}, \mathbb{R}).$$

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by

$$\mathrm{C^1_{rd}} = \mathrm{C^1_{rd}}(\mathbb{T}) = \mathrm{C^1_{rd}}(\mathbb{T}, \mathbb{R}).$$

Exercise 1.59. Are the operators σ, ρ, and μ

(i) continuous;
(ii) rd-continuous;
(iii) regulated?

Some results concerning rd-continuous and regulated functions are contained in the following theorem.

Theorem 1.60. *Assume $f : \mathbb{T} \to \mathbb{R}$.*

(i) *If f is continuous, then f is rd-continuous.*
(ii) *If f is rd-continuous, then f is regulated.*
(iii) *The jump operator σ is rd-continuous.*
(iv) *If f is regulated or rd-continuous, then so is f^σ.*
(v) *Assume f is continuous. If $g : \mathbb{T} \to \mathbb{R}$ is regulated or rd-continuous, then $f \circ g$ has that property too.*

Exercise 1.61. Prove Theorem 1.60.

Definition 1.62. A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called *pre-differentiable* with (region of differentiation) D, provided $D \subset \mathbb{T}^\kappa$, $\mathbb{T}^\kappa \setminus D$ is countable and contains no right-scattered elements of $\mathbb{T}$, and f is differentiable at each $t \in D$.

Example 1.63. Let $\mathbb{T} := \mathbb{P}_{2,1}$ and let $f : \mathbb{T} \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} 0 & \text{if} \quad t \in \cup_{k=0}^{\infty}[3k, 3k+1] \\ t - 3k - 1 & \text{if} \quad t \in [3k+1, 3k+2],\ k \in \mathbb{N}_0. \end{cases}$$

Then f is pre-differentiable with

$$D := \mathbb{T} \setminus \cup_{k=0}^{\infty}\{3k+1\}.$$

Exercise 1.64. For each of the following determine if f is regulated on $\mathbb{T}$, if f is rd-continuous on $\mathbb{T}$, and if f is pre-differentiable. If f is pre-differentiable, find its region of differentiability D.

(i) The function f is defined on a time scale $\mathbb{T}$ and every point $t \in \mathbb{T}$ is isolated.

(ii) Assume $\mathbb{T} = \mathbb{R}$ and

$$f(t) = \begin{cases} 0 & \text{if} \quad t = 0 \\ \frac{1}{t} & \text{if} \quad t \in \mathbb{R} \setminus \{0\}. \end{cases}$$

(iii) Assume $\mathbb{T} = \mathbb{N}_0 \cup \{1 - 1/n : n \in \mathbb{N}\}$ and

$$f(t) = \begin{cases} 0 & \text{if} \quad t \in \mathbb{N} \\ t & \text{otherwise.} \end{cases}$$

(iv) Assume $\mathbb{T} = \mathbb{R}$ and $f(t) = |t|$, $t \in \mathbb{R}$.

(v) Assume $\mathbb{T} = \mathbb{P}_{1,1}$ and

$$f(t) = \begin{cases} 0 & \text{if} \quad t = 2k+1,\ k \in \mathbb{N}_0 \\ t - 2k & \text{if} \quad t \in [2k, 2k+1),\ k \in \mathbb{N}_0. \end{cases}$$

(vi) Assume $\mathbb{T} = \mathbb{P}_{1,1}$ and

$$f(t) = k, \quad t \in [2k, 2k+1], \quad k \in \mathbb{N}_0.$$

Theorem 1.65. *Every regulated function on a compact interval is bounded.*

Proof. Assume $f : [a, b] \to \mathbb{R}$ is unbounded, i.e., for each $n \in \mathbb{N}$ there exists $t_n \in [a, b]$ with $|f(t_n)| > n$. Since

$$\{t_n : n \in \mathbb{N}\} \subset [a, b],$$

there exists a convergent subsequence $\{t_{n_k}\}_{k \in \mathbb{N}}$, i.e.,

$$\lim_{k \to \infty} t_{n_k} = t_0 \quad \text{for some} \quad t_0 \in [a, b]. \tag{1.5}$$

Note that $t_0 \in \mathbb{T}$ since $\{t_{n_k} : k \in \mathbb{N}\} \subset \mathbb{T}$ and $\mathbb{T}$ is closed. By (1.5), t_0 cannot be isolated, and there exists either a subsequence that tends to t_0 from above or a subsequence that tends to t_0 from below, and in any case the limit of $f(t)$ as $t \to t_0$ has to be finite according to regularity, a contradiction. ☐

Remark 1.66. If f is regulated or even if $f \in \mathrm{C_{rd}}$, $\max_{a \le t \le b} f(t)$ and $\min_{a \le t \le b} f(t)$ need not exist. See Exercise 1.64 (iii) for an example of a function which is rd-continuous but does not attain its supremum on $[0, 1]$.

The following mean value theorem holds for pre-differentiable functions and will be used to prove the main existence theorems for pre-antiderivatives and antiderivatives later on in this section. Its proof is an application of the induction principle.

Theorem 1.67 (Mean Value Theorem)**.** *Let f and g be real-valued functions defined on $\mathbb{T}$, both pre-differentiable with D. Then*

$$|f^\Delta(t)| \le g^\Delta(t) \quad \textit{for all} \quad t \in D$$

implies

$$|f(s) - f(r)| \le g(s) - g(r) \quad \textit{for all} \quad r, s \in \mathbb{T},\ r \le s.$$

Proof. Let $r, s \in \mathbb{T}$ with $r \le s$ and denote $[r, s) \setminus D = \{t_n : n \in \mathbb{N}\}$. Let $\varepsilon > 0$. We now show by induction that

$$S(t): \quad |f(t) - f(r)| \le g(t) - g(r) + \varepsilon \left[t - r + \sum_{t_n < t} 2^{-n} \right]$$

holds for all $t \in [r,s]$. Note that once we have shown this, the claim of the mean value theorem follows. We now check the four conditions given in Theorem 1.7.

I. The statement $S(r)$ is trivially satisfied.
II. Let t be right-scattered and assume that $S(t)$ holds. Then

$$\begin{aligned}
|f(\sigma(t)) - f(r)| &= |f(t) + \mu(t) f^\Delta(t) - f(r)| \\
&\le \mu(t)|f^\Delta(t)| + |f(t) - f(r)| \\
&\le \mu(t) g^\Delta(t) + g(t) - g(r) + \varepsilon \left[t - r + \sum_{t_n < t} 2^{-n} \right] \\
&= g(\sigma(t)) - g(r) + \varepsilon \left[t - r + \sum_{t_n < \sigma(t)} 2^{-n} \right] \\
&< g(\sigma(t)) - g(r) + \varepsilon \left[\sigma(t) - r + \sum_{t_n < \sigma(t)} 2^{-n} \right].
\end{aligned}$$

Therefore $S(\sigma(t))$ holds.

III. Suppose $S(t)$ holds and $t \ne s$ is right-dense, i.e., $\sigma(t) = t$. We consider two cases, namely $t \in D$ and $t \notin D$. First of all, suppose $t \in D$. Then f and g are differentiable at t and hence there exists a neighborhood U of t with

$$|f(t) - f(\tau) - f^\Delta(t)(t-\tau)| \le \frac{\varepsilon}{2}|t-\tau| \quad \text{for all} \quad \tau \in U$$

and

$$|g(t) - g(\tau) - g^\Delta(t)(t-\tau)| \le \frac{\varepsilon}{2}|t-\tau| \quad \text{for all} \quad \tau \in U.$$

Thus

$$|f(t) - f(\tau)| \le \left[|f^\Delta(t)| + \frac{\varepsilon}{2} \right] |t-\tau| \quad \text{for all} \quad \tau \in U$$

and

$$g(\tau) - g(t) - g^\Delta(t)(\tau - t) \ge -\frac{\varepsilon}{2}|t-\tau| \quad \text{for all} \quad \tau \in U.$$

Hence we have for all $\tau \in U \cap (t, \infty)$

$$\begin{aligned}
|f(\tau) - f(r)| &\le |f(\tau) - f(t)| + |f(t) - f(r)| \\
&\le \left[|f^\Delta(t)| + \frac{\varepsilon}{2} \right] |t-\tau| + |f(t) - f(r)| \\
&\le \left[g^\Delta(t) + \frac{\varepsilon}{2} \right] |t-\tau| + g(t) - g(r) + \varepsilon \left[t - r + \sum_{t_n < t} 2^{-n} \right] \\
&= g^\Delta(t)(\tau - t) + \frac{\varepsilon}{2}(\tau - t) + g(t) - g(r) + \varepsilon(t-r) + \varepsilon \sum_{t_n < t} 2^{-n} \\
&\le g(\tau) - g(t) + \frac{\varepsilon}{2}|t-\tau| + \frac{\varepsilon}{2}(\tau - t) + g(t) - g(r) \\
&\quad + \varepsilon(t-r) + \varepsilon \sum_{t_n < t} 2^{-n} \\
&= g(\tau) - g(r) + \varepsilon \left[\tau - r + \sum_{t_n < \tau} 2^{-n} \right]
\end{aligned}$$

so that $S(\tau)$ follows for all $\tau \in U \cap (t, \infty)$.

For the second case, suppose $t \notin D$. Then $t = t_m$ for some $m \in \mathbb{N}$. Since f and g are pre-differentiable, they both are continuous and hence there exists a neighborhood U of t with

$$|f(\tau) - f(t)| \leq \frac{\varepsilon}{2} 2^{-m} \quad \text{for all} \quad \tau \in U$$

and

$$|g(\tau) - g(t)| \leq \frac{\varepsilon}{2} 2^{-m} \quad \text{for all} \quad \tau \in U.$$

Therefore

$$g(\tau) - g(t) \geq -\frac{\varepsilon}{2} 2^{-m} \quad \text{for all} \quad \tau \in U$$

and hence

$$\begin{aligned} |f(\tau) - f(r)| &\leq |f(\tau) - f(t)| + |f(t) - f(r)| \\ &\leq \frac{\varepsilon}{2} 2^{-m} + g(t) - g(r) + \varepsilon \left[t - r + \sum_{t_n < t} 2^{-n} \right] \\ &\leq \frac{\varepsilon}{2} 2^{-m} + g(\tau) + \frac{\varepsilon}{2} 2^{-m} - g(r) \\ &\qquad + \varepsilon \left[\tau - r + \sum_{t_n < t} 2^{-n} \right] \\ &= \varepsilon 2^{-m} + g(\tau) - g(r) + \varepsilon \left[\tau - r + \sum_{t_n < t} 2^{-n} \right] \\ &\leq g(\tau) - g(r) + \varepsilon \left[\tau - r + \sum_{t_n < \tau} 2^{-n} \right] \end{aligned}$$

so that again $S(\tau)$ follows for all $\tau \in U \cap (t, \infty)$.

IV. Now let t be left-dense and suppose $S(\tau)$ is true for all $\tau < t$. Then

$$\begin{aligned} \lim_{\tau \to t^-} |f(\tau) - f(r)| &\leq \lim_{\tau \to t^-} \left\{ g(\tau) - g(r) + \varepsilon \left[\tau - r + \sum_{t_n < \tau} 2^{-n} \right] \right\} \\ &\leq \lim_{\tau \to t^-} \left\{ g(\tau) - g(r) + \varepsilon \left[\tau - r + \sum_{t_n < t} 2^{-n} \right] \right\} \end{aligned}$$

implies $S(t)$ as both f and g are continuous at t.

An application of Theorem 1.7 finishes the proof. □

Corollary 1.68. *Suppose f and g are pre-differentiable with D.*

(i) *If U is a compact interval with endpoints $r, s \in \mathbb{T}$, then*

$$|f(s) - f(r)| \leq \left\{ \sup_{t \in U^\kappa \cap D} |f^\Delta(t)| \right\} |s - r|.$$

(ii) *If $f^\Delta(t) = 0$ for all $t \in D$, then f is a constant function.*

(iii) *If $f^\Delta(t) = g^\Delta(t)$ for all $t \in D$, then*

$$g(t) = f(t) + C \quad \textit{for all} \quad t \in \mathbb{T},$$

where C is a constant.

Proof. Suppose f is pre-differentiable with D and let $r, s \in \mathbb{T}$ with $r \leq s$. Define

$$g(t) := \left\{ \sup_{\tau \in [r,s]^\kappa \cap D} |f^\Delta(\tau)| \right\} (t - r) \quad \text{for} \quad t \in \mathbb{T}.$$

Then

$$g^\Delta(t) = \sup_{\tau \in [r,s]^\kappa \cap D} |f^\Delta(\tau)| \geq |f^\Delta(t)| \quad \text{for all} \quad t \in D \cap [r,s]^\kappa.$$

By Theorem 1.67,

$$g(t) - g(r) \geq |f(t) - f(r)| \quad \text{for all} \quad t \in [r,s]$$

so that

$$|f(s) - f(r)| \leq g(s) - g(r) = g(s) = \left\{ \sup_{\tau \in [r,s]^\kappa \cap D} |f^\Delta(\tau)| \right\} (s - r).$$

This completes the proof of part (i). Part (ii) follows immediately from (i), and (iii) follows from (ii). □

Exercise 1.69. Prove Theorem 1.68 (ii) and (iii).

The main existence theorem for pre-antiderivatives now reads as follows. We will prove this theorem in a more general form in Chapter 8.

Theorem 1.70 (Existence of Pre-Antiderivatives). *Let f be regulated. Then there exists a function F which is pre-differentiable with region of differentiation D such that*

$$F^\Delta(t) = f(t) \quad \textit{holds for all} \quad t \in D.$$

Proof. See the proof of Theorem 8.13. □

Definition 1.71. Assume $f : \mathbb{T} \to \mathbb{R}$ is a regulated function. Any function F as in Theorem 1.70 is called a *pre-antiderivative* of f. We define the *indefinite integral* of a regulated function f by

$$\int f(t)\Delta t = F(t) + C,$$

where C is an arbitrary constant and F is a pre-antiderivative of f. We define the *Cauchy integral* by

$$\int_r^s f(t)\Delta t = F(s) - F(r) \quad \text{for all} \quad r, s \in \mathbb{T}.$$

A function $F : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ provided

$$F^\Delta(t) = f(t) \quad \text{holds for all} \quad t \in \mathbb{T}^\kappa.$$

Example 1.72. If $\mathbb{T} = \mathbb{Z}$, evaluate the indefinite integral

$$\int a^t \Delta t,$$

where $a \neq 1$ is a constant. Since

$$\left[\frac{a^t}{a-1}\right]^\Delta = \Delta \left[\frac{a^t}{a-1}\right] = \frac{a^{t+1} - a^t}{a - 1} = a^t,$$

we get that

$$\int a^t \Delta t = \frac{a^t}{a-1} + C,$$

where C is an arbitrary constant.

Exercise 1.73. Show that if $\mathbb{T} = \mathbb{Z}$, $k \neq -1$, and $\alpha \in \mathbb{R}$, then

(i) $\int (t+\alpha)^{(k)} \Delta t = \frac{(t+\alpha)^{(k+1)}}{k+1} + C$;
(ii) $\int \binom{t}{\alpha} \Delta \tau = \binom{t}{\alpha+1} + C$.

Theorem 1.74 (Existence of Antiderivatives)**.** *Every rd-continuous function has an antiderivative. In particular if $t_0 \in \mathbb{T}$, then F defined by*

$$F(t) := \int_{t_0}^{t} f(\tau)\Delta\tau \quad \textit{for} \quad t \in \mathbb{T}$$

is an antiderivative of f.

Proof. Suppose f is an rd-continuous function. By Theorem 1.60 (ii), f is regulated. Let F be a function guaranteed to exist by Theorem 1.70, together with D, satisfying $F(t_0) = x_0$ and

$$F^{\Delta}(t) = f(t) \quad \text{for all} \quad t \in D.$$

This F is pre-differentiable with D. We have to show that $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^{\kappa}$ (this, of course, includes all points in $\mathbb{T}^{\kappa} \setminus D$). So let $t \in \mathbb{T}^{\kappa} \setminus D$. Then t is right-dense because $\mathbb{T}^{\kappa} \setminus D$ cannot contain any right-scattered points according to Definition 1.62. Since f is rd-continuous, it is continuous at t. Let $\varepsilon > 0$. Then there exists a neighborhood U of t with

$$|f(s) - f(t)| \le \varepsilon \quad \text{for all} \quad s \in U.$$

Define

$$h(\tau) := F(\tau) - f(t)(\tau - t_0) \quad \text{for} \quad \tau \in \mathbb{T}.$$

Then h is pre-differentiable with D and we have

$$h^{\Delta}(\tau) = F^{\Delta}(\tau) - f(t) = f(\tau) - f(t) \quad \text{for all} \quad \tau \in D.$$

Hence

$$|h^{\Delta}(s)| = |f(s) - f(t)| \le \varepsilon \quad \text{for all} \quad s \in D \cap U.$$

Therefore

$$\sup_{s \in D\cap U} |h^{\Delta}(s)| \le \varepsilon.$$

Thus, by Corollary 1.68, we have for $r \in U$

$$\begin{aligned} |F(t) - F(r) - f(t)(t-r)| &= |h(t) + f(t)(t-t_0) - [h(r) + f(t)(r-t_0)] \\ &\quad -f(t)(t-r)| \\ &= |h(t) - h(r)| \\ &\le \left\{ \sup_{s\in D\cap U} |h^{\Delta}(s)| \right\} |t-r| \\ &\le \varepsilon |t-r|. \end{aligned}$$

But this shows that F is differentiable at t with $F^{\Delta}(t) = f(t)$. ☐

Table 1.3. The two most important Examples

Time scale $\mathbb{T}$	$\mathbb{R}$	$\mathbb{Z}$
Backward jump operator $\rho(t)$	t	$t-1$
Forward jump operator $\sigma(t)$	t	$t+1$
Graininess $\mu(t)$	0	1
Derivative $f^{\Delta}(t)$	$f'(t)$	$\Delta f(t)$
Integral $\int_a^b f(t)\Delta t$	$\int_a^b f(t)dt$	$\sum_{t=a}^{b-1} f(t)$ (if $a<b$)
Rd-continuous f	continuous f	any f

Theorem 1.75. *If $f \in \mathrm{C_{rd}}$ and $t \in \mathbb{T}^{\kappa}$, then*

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t).$$

Proof. By Theorem 1.74, there exists an antiderivative F of f, and

$$\begin{aligned}\int_t^{\sigma(t)} f(\tau)\Delta\tau &= F(\sigma(t)) - F(t)\\ &= \mu(t)F^{\Delta}(t)\\ &= \mu(t)f(t),\end{aligned}$$

where the second equation holds because of Theorem 1.16 (iv). □

Theorem 1.76. *If $f^{\Delta} \geq 0$, then f is increasing.*

Proof. Let $f^{\Delta} \geq 0$ on $[a,b]$ and let $s,t \in \mathbb{T}$ with $a \leq s \leq t \leq b$. Then

$$f(t) = f(s) + \int_s^t f^{\Delta}(\tau)\Delta\tau \geq f(s)$$

so that the conclusion follows. □

Theorem 1.77. *If $a,b,c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and $f,g \in \mathrm{C_{rd}}$, then*

(i) $\int_a^b [f(t)+g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$;
(ii) $\int_a^b \alpha f(t)\Delta t = \alpha \int_a^b f(t)\Delta t$;
(iii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$;
(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$;
(v) $\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(t)\Delta t$;
(vi) $\int_a^b f(t)g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t$;
(vii) $\int_a^a f(t)\Delta t = 0$;

(viii) *if* $|f(t)| \leq g(t)$ *on* $[a, b)$, *then*

$$\left| \int_a^b f(t)\Delta t \right| \leq \int_a^b g(t)\Delta t;$$

(ix) *if* $f(t) \geq 0$ *for all* $a \leq t < b$, *then* $\int_a^b f(t)\Delta t \geq 0$.

Proof. These results follow easily from Definition 1.71, Theorem 1.20, and Theorem 1.67. We only prove (i), (iv), and (v), and leave the rest of the proof as an exercise (see Exercise 1.78). Since f and g are rd-continuous, they possess antiderivatives F and G by Theorem 1.74. By Theorem 1.20 (i), $F + G$ is an antiderivative of $f + g$ so that

$$\begin{aligned}
\int_a^b (f+g)(t)\Delta t &= (F+G)(b) - (F+G)(a) \\
&= F(b) - F(a) + G(b) - G(a) \\
&= \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t.
\end{aligned}$$

Also

$$\begin{aligned}
\int_a^b f(t)\Delta t &= F(b) - F(a) \\
&= F(c) - F(a) + F(b) - F(c) \\
&= \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t.
\end{aligned}$$

Finally, since fg is an antiderivative of $f^\sigma g^\Delta + f^\Delta g$,

$$\int_a^b [f^\sigma g^\Delta + f^\Delta g](t)\Delta t = (fg)(b) - (fg)(a),$$

so that (v) follows by using (i). □

Note that the formulas in Theorem 1.77 (v) and (vi) are called *integration by parts* formulas. Also note that all of the formulas given in Theorem 1.77 also hold for the case that f and g are only regulated functions.

Exercise 1.78. Finish the proof of Theorem 1.77. Also prove each item of Theorem 1.77 assuming that the functions f and g are merely regulated rather than rd-continuous.

Theorem 1.79. *Let* $a, b \in \mathbb{T}$ *and* $f \in \mathrm{C}_{\mathrm{rd}}$.

(i) *If* $\mathbb{T} = \mathbb{R}$, *then*

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt,$$

where the integral on the right is the usual Riemann integral from calculus.

(ii) *If* $[a, b]$ *consists of only isolated points, then*

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t\in[a,b)} \mu(t)f(t) & \text{if} \quad a < b \\ 0 & \text{if} \quad a = b \\ -\sum_{t\in[b,a)} \mu(t)f(t) & \text{if} \quad a > b. \end{cases}$$

(iii) *If* $\mathbb{T} = h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$, *where* $h > 0$, *then*

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h & \text{if} \quad a < b \\ 0 & \text{if} \quad a = b \\ -\sum_{k=\frac{b}{h}}^{\frac{a}{h}-1} f(kh)h & \text{if} \quad a > b. \end{cases}$$

(iv) *If* $\mathbb{T} = \mathbb{Z}$, *then*

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t=a}^{b-1} f(t) & \text{if} \quad a < b \\ 0 & \text{if} \quad a = b \\ -\sum_{t=b}^{a-1} f(t) & \text{if} \quad a > b. \end{cases}$$

Proof. Part (i) follows from Example 1.18 (i) and the standard fundamental theorem of calculus. We now prove (ii). First note that $[a, b]$ contains only finitely many points since each point in $[a, b]$ is isolated. Assume that $a < b$ and let $[a, b] = \{t_0, t_1, \dots, t_n\}$, where

$$a = t_0 < t_1 < t_2 < \cdots < t_n = b.$$

Then

$$\begin{aligned} \int_a^b f(t)\Delta t &= \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} f(t)\Delta t \\ &= \sum_{i=0}^{n-1} \int_{t_i}^{\sigma(t_i)} f(t)\Delta t \\ &= \sum_{i=0}^{n-1} \mu(t_i) f(t_i) \\ &= \sum_{t\in[a,b)} \mu(t) f(t), \end{aligned}$$

where the third equation above follows from Theorem 1.75. If $b < a$, then the result follows from what we just proved and Theorem 1.77 (iii). If $a = b$, then $\int_a^b f(t)\Delta t = 0$ by Theorem 1.77 (vii). Parts (iii) and (iv) are special cases of (ii) (see Exercise 1.80). □

Exercise 1.80. Prove that Theorem 1.79 (iii) and (iv) follow from Theorem 1.79 (ii).

Exercise 1.81. Let $a \in \mathbb{T}$, where $\mathbb{T}$ is an arbitrary time scale and evaluate $\int_a^t 1\Delta s$. Also evaluate $\int_0^t s\Delta s$ for $t \in \mathbb{T}$, for $\mathbb{T} = \mathbb{R}$, for $\mathbb{T} = \mathbb{Z}$, for $\mathbb{T} = h\mathbb{Z}$, and for $\mathbb{T} = [0, 1] \cup [2, 3]$.

We next define the improper integral $\int_a^\infty f(t)\Delta t$ as one would expect.

Definition 1.82. If $a \in \mathbb{T}$, $\sup \mathbb{T} = \infty$, and f is rd-continuous on $[a, \infty)$, then we define the improper integral by

$$\int_a^\infty f(t)\Delta t := \lim_{b\to\infty} \int_a^b f(t)\Delta t$$

provided this limit exists, and we say that the improper integral *converges* in this case. If this limit does not exist, then we say that the improper integral *diverges*.

We now give two exercises concerning improper integrals.

Exercise 1.83. Evaluate the integral

$$\int_1^\infty \frac{1}{t^2}\Delta t$$

if $\mathbb{T} = q^{\mathbb{N}_0}$, where $q > 1$.

Exercise 1.84. Assume $a \in \mathbb{T}$, $a > 0$ and $\sup \mathbb{T} = \infty$. Evaluate

$$\int_a^\infty \frac{1}{t\sigma(t)}\Delta t.$$

1.5. Chain Rules

If $f, g : \mathbb{R} \to \mathbb{R}$, then the chain rule from calculus is that if g is differentiable at t and if f is differentiable at $g(t)$, then

$$(f \circ g)'(t) = f'(g(t))\, g'(t).$$

The next example shows that the chain rule as we know it in calculus does not hold for all time scales.

Example 1.85. Assume $f, g : \mathbb{Z} \to \mathbb{Z}$ are defined by $f(t) = t^2$, $g(t) = 2t$ and our time scale is $\mathbb{T} = \mathbb{Z}$. It is easy to see that

$$(f \circ g)^\Delta(t) = 8t + 4 \neq 8t + 2 = f^\Delta(g(t))\, g^\Delta(t) \quad \text{for all} \quad t \in \mathbb{Z}.$$

Hence the chain rule as we know it in calculus does not hold in this setting.

Exercise 1.86. Assume that $\mathbb{T} = \mathbb{Z}$ and $f(t) = g(t) = t^2$. Show that for all $t \neq 0$

$$(f \circ g)^\Delta(t) \neq f^\Delta(g(t))\, g^\Delta(t).$$

Sometimes the following substitute of the "continuous" chain rule is useful.

Theorem 1.87 (Chain Rule)**.** *Assume $g : \mathbb{R} \to \mathbb{R}$ is continuous, $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable on $\mathbb{T}^\kappa$, and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then there exists c in the real interval $[t, \sigma(t)]$ with*

$$(f \circ g)^\Delta(t) = f'(g(c))\, g^\Delta(t). \tag{1.6}$$

Proof. Fix $t \in \mathbb{T}^\kappa$. First we consider the case where t is right-scattered. In this case

$$(f \circ g)^\Delta(t) = \frac{f(g(\sigma(t))) - f(g(t))}{\mu(t)}.$$

If $g(\sigma(t)) = g(t)$, then we get $(f \circ g)^\Delta(t) = 0$ and $g^\Delta(t) = 0$ and so (1.6) holds for any c in the real interval $[t, \sigma(t)]$. Hence we can assume $g(\sigma(t)) \neq g(t)$. Then

$$\begin{aligned}(f \circ g)^\Delta(t) &= \frac{f(g(\sigma(t))) - f(g(t))}{g(\sigma(t)) - g(t)} \cdot \frac{g(\sigma(t)) - g(t)}{\mu(t)} \\ &= f'(\xi) g^\Delta(t)\end{aligned}$$

by the mean value theorem, where ξ is between $g(t)$ and $g(\sigma(t))$. Since $g : \mathbb{R} \to \mathbb{R}$ is continuous, there is $c \in [t, \sigma(t)]$ such that $g(c) = \xi$, which gives us the desired result.

It remains to consider the case when t is right-dense. In this case

$$\begin{aligned}(f\circ g)^{\Delta}(t) &= \lim_{s\to t}\frac{f(g(t))-f(g(s))}{t-s}\\ &= \lim_{s\to t}\left\{f'(\xi_s)\cdot\frac{g(t)-g(s)}{t-s}\right\}\end{aligned}$$

by the mean value theorem in calculus, where ξ_s is between $g(s)$ and $g(t)$. By the continuity of g we get that $\lim_{s\to t}\xi_s = g(t)$ which gives us the desired result. □

Example 1.88. Given $\mathbb{T} = \mathbb{Z}$, $f(t) = t^2$, $g(t) = 2t$, find directly the value c guaranteed by Theorem 1.87 so that

$$(f\circ g)^{\Delta}(3) = f'(g(c))\, g^{\Delta}(3)$$

and show that c is in the interval guaranteed by Theorem 1.87. Using the calculations we made in Example 1.85 we get that this last equation becomes

$$28 = (4c)2.$$

Solving for c we get that $c = \frac{7}{2}$ which is in the real interval $[3, \sigma(3)] = [3, 4]$ as we are guaranteed by Theorem 1.87.

Exercise 1.89. Assume that $\mathbb{T} = \mathbb{Z}$ and $f(t) = g(t) = t^2$. Find directly the c guaranteed by Theorem 1.87 so that $(f\circ g)^{\Delta}(2) = f'(g(c))\, g^{\Delta}(2)$ and be sure to note that c is in the interval guaranteed by Theorem 1.87.

Now we present a chain rule which calculates $(f\circ g)^{\Delta}$, where

$$g:\mathbb{T}\to\mathbb{R} \quad\text{and}\quad f:\mathbb{R}\to\mathbb{R}.$$

This chain rule is due to Christian Pötzsche, who derived it first in 1998 (see also Stefan Keller's PhD thesis [**190**] and [**234**]).

Theorem 1.90 (Chain Rule). *Let $f:\mathbb{R}\to\mathbb{R}$ be continuously differentiable and suppose $g:\mathbb{T}\to\mathbb{R}$ is delta differentiable. Then $f\circ g:\mathbb{T}\to\mathbb{R}$ is delta differentiable and the formula*

$$(f\circ g)^{\Delta}(t) = \left\{\int_0^1 f'\left(g(t) + h\mu(t)g^{\Delta}(t)\right)dh\right\} g^{\Delta}(t)$$

holds.

Proof. First of all we apply the ordinary substitution rule from calculus to find

$$\begin{aligned}f(g(\sigma(t))) - f(g(s)) &= \int_{g(s)}^{g(\sigma(t))} f'(\tau)d\tau\\ &= [g(\sigma(t)) - g(s)]\int_0^1 f'(hg(\sigma(t)) + (1-h)g(s))dh.\end{aligned}$$

Let $t\in\mathbb{T}^{\kappa}$ and $\varepsilon > 0$ be given. Since g is differentiable at t, there exists a neighborhood U_1 of t such that

$$|g(\sigma(t)) - g(s) - g^{\Delta}(t)(\sigma(t) - s)| \le \varepsilon^*|\sigma(t) - s|,$$

where

$$\varepsilon^* = \frac{\varepsilon}{1 + 2\int_0^1 |f'(hg(\sigma(t)) + (1-h)g(t))|dh}$$

for all $s \in U_1$. Moreover, f' is continuous on $\mathbb{R}$, and therefore it is uniformly continuous on closed subsets of $\mathbb{R}$, and (observe also that g is continuous as it is differentiable, see Theorem 1.16 (i)) hence there exists a neighborhood U_2 of t such that

$$|f'(hg(\sigma(t)) + (1-h)g(s)) - f'(hg(\sigma(t)) + (1-h)g(t))| \leq \frac{\varepsilon}{2(\varepsilon^* + |g^\Delta(t)|)}$$

for all $s \in U_2$. To see this, note also that

$$\begin{aligned} |hg(\sigma(t)) + (1-h)g(s) - (hg(\sigma(t)) + (1-h)g(t))| &= (1-h)|g(s) - g(t)| \\ &\leq |g(s) - g(t)| \end{aligned}$$

holds for all $0 \leq h \leq 1$. We then define $U = U_1 \cap U_2$ and let $s \in U$. For convenience we put

$$\alpha = hg(\sigma(t)) + (1-h)g(s) \quad \text{and} \quad \beta = hg(\sigma(t)) + (1-h)g(t).$$

Then we have

$$\begin{aligned} &\left|(f \circ g)(\sigma(t)) - (f \circ g)(s) - (\sigma(t) - s)g^\Delta(t) \int_0^1 f'(\beta)dh\right| \\ = \;& \left|[g(\sigma(t)) - g(s)] \int_0^1 f'(\alpha)dh - (\sigma(t) - s)g^\Delta(t) \int_0^1 f'(\beta)dh\right| \\ = \;& \left|[g(\sigma(t)) - g(s) - (\sigma(t) - s)g^\Delta(t)] \int_0^1 f'(\alpha)dh \right. \\ &\quad \left. + (\sigma(t) - s)g^\Delta(t) \int_0^1 (f'(\alpha) - f'(\beta))dh\right| \\ \leq \;& |g(\sigma(t)) - g(s) - (\sigma(t) - s)g^\Delta(t)| \int_0^1 |f'(\alpha)|dh \\ &\quad + |\sigma(t) - s||g^\Delta(t)| \int_0^1 |f'(\alpha) - f'(\beta)|dh \\ \leq \;& \varepsilon^*|\sigma(t) - s| \int_0^1 |f'(\alpha)|dh + |\sigma(t) - s||g^\Delta(t)| \int_0^1 |f'(\alpha) - f'(\beta)|dh \\ \leq \;& \varepsilon^*|\sigma(t) - s| \int_0^1 |f'(\beta)|dh + [\varepsilon^* + |g^\Delta(t)|]|\sigma(t) - s| \int_0^1 |f'(\alpha) - f'(\beta)|dh \\ = \;& \frac{\varepsilon}{2}|\sigma(t) - s| + \frac{\varepsilon}{2}|\sigma(t) - s| \\ = \;& \varepsilon|\sigma(t) - s|. \end{aligned}$$

Therefore $f \circ g$ is differentiable at t and the derivative is as claimed above. □

Example 1.91. We define $g : \mathbb{Z} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ by

$$g(t) = t^2 \quad \text{and} \quad f(x) = \exp(x).$$

Then

$$g^\Delta(t) = (t+1)^2 - t^2 = 2t + 1 \quad \text{and} \quad f'(x) = \exp(x).$$

Hence we have by Theorem 1.90

$$\begin{aligned}(f\circ g)^{\Delta}(t) &= \left\{\int_0^1 f'\left(g(t)+h\mu(t)g^{\Delta}(t)\right)dh\right\}g^{\Delta}(t)\\ &= (2t+1)\int_0^1 \exp(t^2+h(2t+1))dh\\ &= (2t+1)\exp(t^2)\int_0^1 \exp(h(2t+1))dh\\ &= (2t+1)\exp(t^2)\frac{1}{2t+1}\left[\exp(h(2t+1))\right]_{h=0}^{h=1}\\ &= (2t+1)\exp(t^2)\frac{1}{2t+1}(\exp(2t+1)-1)\\ &= \exp(t^2)(\exp(2t+1)-1).\end{aligned}$$

On the other hand, it is easy to check that we have indeed

$$\begin{aligned}\Delta f(g(t)) &= f(g(t+1))-f(g(t))\\ &= \exp((t+1)^2)-\exp(t^2)\\ &= \exp(t^2+2t+1)-\exp(t^2)\\ &= \exp(t^2)(\exp(2t+1)-1).\end{aligned}$$

In the remainder of this section we present some results related to results in the paper by C. D. Ahlbrandt, M. Bohner, and J. Ridenhour [**24**]. Let $\mathbb{T}$ be a time scale and $\nu:\mathbb{T}\to\mathbb{R}$ be a strictly increasing function such that $\tilde{\mathbb{T}}=\nu(\mathbb{T})$ is also a time scale. By $\tilde{\sigma}$ we denote the jump function on $\tilde{\mathbb{T}}$ and by $^{\tilde{\Delta}}$ we denote the derivative on $\tilde{\mathbb{T}}$. Then $\nu\circ\sigma=\tilde{\sigma}\circ\nu$.

Exercise 1.92. Prove that $\nu\circ\sigma=\tilde{\sigma}\circ\nu$ under the hypotheses of the above paragraph.

Theorem 1.93 (Chain Rule)**.** *Assume $\nu:\mathbb{T}\to\mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}}:=\nu(\mathbb{T})$ is a time scale. Let $w:\tilde{\mathbb{T}}\to\mathbb{R}$. If $\nu^{\Delta}(t)$ and $w^{\tilde{\Delta}}(\nu(t))$ exist for $t\in\mathbb{T}^{\kappa}$, then*

$$(w\circ\nu)^{\Delta}=(w^{\tilde{\Delta}}\circ\nu)\nu^{\Delta}.$$

Proof. Let $0<\varepsilon<1$ be given and define $\varepsilon^*=\varepsilon\left[1+|\nu^{\Delta}(t)|+|w^{\tilde{\Delta}}(\nu(t))|\right]^{-1}$. Note that $0<\varepsilon^*<1$. According to the assumptions, there exist neighborhoods $\mathcal{N}_1$ of t and $\mathcal{N}_2$ of $\nu(t)$ such that

$$|\nu(\sigma(t))-\nu(s)-(\sigma(t)-s)\nu^{\Delta}(t)|\le\varepsilon^*|\sigma(t)-s|\quad\text{for all}\quad s\in\mathcal{N}_1$$

and

$$|w(\tilde{\sigma}(\nu(t)))-w(r)-(\tilde{\sigma}(\nu(t))-r)w^{\tilde{\Delta}}(\nu(t))|\le\varepsilon^*|\tilde{\sigma}(\nu(t))-r|,\ r\in\mathcal{N}_2.$$

Put $\mathcal{N} = \mathcal{N}_1 \cap \nu^{-1}(\mathcal{N}_2)$ and let $s \in \mathcal{N}$. Then $s \in \mathcal{N}_1$ and $\nu(s) \in \mathcal{N}_2$ and

$$\begin{aligned}
&|w(\nu(\sigma(t))) - w(\nu(s)) - (\sigma(t) - s)[w^{\tilde{\Delta}}(\nu(t))\nu^{\Delta}(t)]| \\
&= |w(\nu(\sigma(t))) - w(\nu(s)) - (\tilde{\sigma}(\nu(t)) - \nu(s))w^{\tilde{\Delta}}(\nu(t)) \\
&\quad +[\tilde{\sigma}(\nu(t)) - \nu(s) - (\sigma(t) - s)\nu^{\Delta}(t)]w^{\tilde{\Delta}}(\nu(t))| \\
&\le \varepsilon^*|\tilde{\sigma}(\nu(t)) - \nu(s)| + \varepsilon^*|\sigma(t) - s||w^{\tilde{\Delta}}(\nu(t))| \\
&\le \varepsilon^* \left\{|\tilde{\sigma}(\nu(t)) - \nu(s) - (\sigma(t) - s)\nu^{\Delta}(t)| + |\sigma(t) - s||\nu^{\Delta}(t)| \right. \\
&\quad \left. +|\sigma(t) - s||w^{\tilde{\Delta}}(\nu(t))|\right\} \\
&\le \varepsilon^* \left\{\varepsilon^*|\sigma(t) - s| + |\sigma(t) - s||\nu^{\Delta}(t)| + |\sigma(t) - s||w^{\tilde{\Delta}}(\nu(t))|\right\} \\
&= \varepsilon^*|\sigma(t) - s| \left\{\varepsilon^* + |\nu^{\Delta}(t)| + |w^{\tilde{\Delta}}(\nu(t))|\right\} \\
&\le \varepsilon^* \left\{1 + |\nu^{\Delta}(t)| + |w^{\tilde{\Delta}}(\nu(t))|\right\} |\sigma(t) - s| \\
&= \varepsilon|\sigma(t) - s|.
\end{aligned}$$

This proves the claim. ☐

Example 1.94. Let $\mathbb{T} = \mathbb{N}_0$ and $\nu(t) = 4t + 1$. Hence

$$\tilde{\mathbb{T}} = \nu(\mathbb{T}) = \{4n + 1 : \ n \in \mathbb{N}_0\} = \{1, 5, 9, 13, \dots\}.$$

Moreover, let $w : \tilde{\mathbb{T}} \to \mathbb{R}$ be defined by $w(t) = t^2$. Then

$$(w \circ \nu)(t) = w(\nu(t)) = w(4t + 1) = (4t + 1)^2$$

and hence

$$\begin{aligned}
(w \circ \nu)^{\Delta}(t) &= [4(t + 1) + 1]^2 - (4t + 1)^2 \\
&= (4t + 5)^2 - (4t + 1)^2 \\
&= 16t^2 + 40t + 25 - 16t^2 - 8t - 1 \\
&= 32t + 24.
\end{aligned}$$

Now we apply Theorem 1.93 to obtain the derivative of this composite function. We first calculate $\nu^{\Delta}(t) \equiv 4$ and then

$$w^{\tilde{\Delta}}(t) = \frac{w(\tilde{\sigma}(t)) - w(t)}{\tilde{\sigma}(t) - t} = \frac{(t + 4)^2 - t^2}{t + 4 - t} = \frac{8t + 16}{4} = 2t + 4$$

and therefore

$$(w^{\tilde{\Delta}} \circ \nu)(t) = w^{\tilde{\Delta}}(\nu(t)) = w^{\tilde{\Delta}}(4t + 1) = 2(4t + 1) + 4 = 8t + 6.$$

Thus we obtain

$$[(w^{\tilde{\Delta}} \circ \nu)\nu^{\Delta}](t) = (8t + 6)4 = 32t + 24 = (w \circ \nu)^{\Delta}(t).$$

Exercise 1.95. Let $\mathbb{T} = \mathbb{N}_0$, $\nu(t) = t^2$, $\tilde{\mathbb{T}} = \nu(\mathbb{T})$, and $w(t) = 2t^2 + 3$. Show directly as in Example 1.94 that

$$(w \circ \nu)^{\Delta} = (w^{\tilde{\Delta}} \circ \nu)\nu^{\Delta}.$$

Exercise 1.96. Find a time scale $\mathbb{T}$ and a strictly increasing function $\nu : \mathbb{T} \to \mathbb{R}$ such that $\nu(\mathbb{T})$ is not a time scale.

As a consequence of the above Theorem 1.93 we can now write down a formula for the derivative of the inverse function.

Theorem 1.97 (Derivative of the Inverse). *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Then*

$$\frac{1}{\nu^{\Delta}} = (\nu^{-1})^{\tilde{\Delta}} \circ \nu$$

at points where ν^{Δ} is different from zero.

Proof. Let $w = \nu^{-1} : \tilde{\mathbb{T}} \to \mathbb{T}$ in the previous theorem. □

Another consequence of Theorem 1.93 is the substitution rule for integrals.

Theorem 1.98 (Substitution). *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is an rd-continuous function and ν is differentiable with rd-continuous derivative, then if $a, b \in \mathbb{T}$*

$$\int_a^b f(t)\nu^{\Delta}(t)\Delta t = \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s)\tilde{\Delta} s.$$

Proof. Since $f\nu^{\Delta}$ is an rd-continuous function, it possesses an antiderivative F by Theorem 1.74, i.e., $F^{\Delta} = f\nu^{\Delta}$, and

$$\begin{aligned}
\int_a^b f(t)\nu^{\Delta}(t)\Delta t &= \int_a^b F^{\Delta}(t)\Delta t \\
&= F(b) - F(a) \\
&= (F \circ \nu^{-1})(\nu(b)) - (F \circ \nu^{-1})(\nu(a)) \\
&= \int_{\nu(a)}^{\nu(b)} (F \circ \nu^{-1})^{\tilde{\Delta}}(s)\tilde{\Delta} s \\
&= \int_{\nu(a)}^{\nu(b)} (F^{\Delta} \circ \nu^{-1})(s)(\nu^{-1})^{\tilde{\Delta}}(s)\tilde{\Delta} s \\
&= \int_{\nu(a)}^{\nu(b)} ((f\nu^{\Delta}) \circ \nu^{-1})(s)(\nu^{-1})^{\tilde{\Delta}}(s)\tilde{\Delta} s \\
&= \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s)[(\nu^{\Delta} \circ \nu^{-1})(\nu^{-1})^{\tilde{\Delta}}](s)\tilde{\Delta} s \\
&= \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s)\tilde{\Delta} s,
\end{aligned}$$

where for the fifth equal sign we have used Theorem 1.93 and in the last step we have used Theorem 1.97. □

Example 1.99. In this example we use the method of substitution (Theorem 1.98) to evaluate the integral

$$\int_0^t \left(\sqrt{\tau^2 + 1} + \tau\right) 3^{\tau^2} \Delta\tau$$

for $t \in \mathbb{T} := \mathbb{N}_0^{\frac{1}{2}} = \{\sqrt{n} : n \in \mathbb{N}_0\}$. We take

$$\nu(t) = t^2$$

for $t \in \mathbb{N}_0^{\frac{1}{2}}$. Then $\nu : \mathbb{N}_0^{\frac{1}{2}} \to \mathbb{R}$ is strictly increasing and $\nu\left(\mathbb{N}_0^{\frac{1}{2}}\right) = \mathbb{N}_0$ is a time scale. From Exercise 1.19 we get that

$$\nu^\Delta(t) = \sqrt{t^2+1} + t.$$

Hence if $f(t) := 3^{t^2}$, we get from Theorem 1.98 that

$$\begin{aligned}\int_0^t \left(\sqrt{\tau^2+1}+\tau\right) 3^{\tau^2}\Delta\tau &= \int_0^t f(\tau)\nu^\Delta(\tau)\Delta\tau \\ &= \int_0^{t^2} f(\sqrt{s})\tilde{\Delta} s \\ &= \int_0^{t^2} 3^s \tilde{\Delta} s \\ &= \left[\frac{1}{2}3^s\right]_0^{t^2} \\ &= \frac{1}{2}(3^{t^2}-1).\end{aligned}$$

Exercise 1.100. Evaluate the integral

$$\int_0^t 2\tau(2\tau-1)\Delta\tau$$

for $t \in \mathbb{T} := \{\frac{n}{2} : n \in \mathbb{N}_0\}$ by applying Theorem 1.98 with $\nu(t) = 2t$.

Exercise 1.101. Evaluate the integral

$$\int_0^t \left[(\tau^3+1)^{\frac{2}{3}} + \tau(\tau^3+1)^{\frac{1}{3}} + \tau^2\right] 2^{\tau^3}\Delta\tau$$

for $t \in \mathbb{T} := \{\sqrt[3]{n} : n \in \mathbb{N}_0\}$.

1.6. Polynomials

An antiderivative of 0 is 1, an antiderivative of 1 is t, but it is not possible to find a closed formula (for an arbitrary time scale) of an antiderivative of t. Certainly t^2 is not the solution, as the derivative of t^2 is

$$t + \sigma(t)$$

which is, as we know, e.g., by Example 1.56, not even necessarily a differentiable function (although it is the product of two differentiable functions). Similarly, none of the "classical" polynomials are necessarily more than once differentiable, see Theorem 1.24. So the question arises which function plays the rôle of e.g., t^2, in the time scales calculus. It could be either

$$\int_0^t \sigma(\tau)\Delta\tau \quad \text{or} \quad \int_0^t \tau\Delta\tau.$$

In fact, if we define

$$g_2(t,s) = \int_s^t (\sigma(\tau)-s)\Delta\tau \quad \text{and} \quad h_2(t,s) = \int_s^t (\tau-s)\Delta\tau,$$

we find the following relation between g_2 and h_2:

$$\begin{aligned} g_2(t,s) &= \int_s^t (\sigma(\tau)-s)\Delta\tau \\ &= \int_s^t (\sigma(\tau)+\tau)\Delta\tau - \int_s^t \tau\Delta\tau - \int_s^t s\Delta\tau \\ &= \int_s^t (\tau^2)^\Delta \Delta\tau + \int_t^s \tau\Delta\tau - s(t-s) \\ &= \int_t^s \tau\Delta\tau + t^2 - s^2 - s(t-s) \\ &= \int_t^s (\tau - t)\Delta\tau \\ &= h_2(s,t). \end{aligned}$$

In this section we give a Taylor's formula for functions on a general time scale. Many of the results in this section can be found in R. P. Agarwal and M. Bohner [**9**]. The generalized polynomials, that also occur in Taylor's formula, are the functions $g_k, h_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$, defined recursively as follows: The functions g_0 and h_0 are

$$g_0(t,s) = h_0(t,s) = 1 \quad \text{for all} \quad s,t \in \mathbb{T}, \tag{1.7}$$

and, given g_k and h_k for $k \in \mathbb{N}_0$, the functions g_{k+1} and h_{k+1} are

$$g_{k+1}(t,s) = \int_s^t g_k(\sigma(\tau),s)\Delta\tau \quad \text{for all} \quad s,t \in \mathbb{T} \tag{1.8}$$

and

$$h_{k+1}(t,s) = \int_s^t h_k(\tau,s)\Delta\tau \quad \text{for all} \quad s,t \in \mathbb{T}. \tag{1.9}$$

Note that the functions g_k and h_k are all well defined according to Theorem 1.60 and Theorem 1.74. If we let $h_k^\Delta(t,s)$ denote for each fixed s the delta derivative of $h_k(t,s)$ with respect to t, then

$$h_k^\Delta(t,s) = h_{k-1}(t,s) \quad \text{for} \quad k \in \mathbb{N},\ t \in \mathbb{T}^\kappa.$$

Similarly

$$g_k^\Delta(t,s) = g_{k-1}(\sigma(t),s) \quad \text{for} \quad k \in \mathbb{N},\ t \in \mathbb{T}^\kappa.$$

The above definitions obviously imply

$$g_1(t,s) = h_1(t,s) = t - s \quad \text{for all} \quad s,t \in \mathbb{T}.$$

However, finding g_k and h_k for $k > 1$ is not easy in general. But for a particular given time scale it might be easy to find these functions. We will consider several examples first before we present Taylor's formula in general.

Example 1.102. For the cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ it is easy to find the functions g_k and h_k:

First, consider $\mathbb{T} = \mathbb{R}$. Then $\sigma(t) = t$ for $t \in \mathbb{R}$ so that $g_k = h_k$ for $k \in \mathbb{N}_0$. We have

$$\begin{aligned} g_2(t,s) &= h_2(t,s) = \int_s^t (\tau - s)d\tau \\ &= \left.\frac{(\tau - s)^2}{2}\right|_{\tau=s}^{\tau=t} \\ &= \frac{(t-s)^2}{2}. \end{aligned}$$

We claim that for $k \in \mathbb{N}_0$

$$g_k(t,s) = h_k(t,s) = \frac{(t-s)^k}{k!} \quad \text{for all} \quad s,t \in \mathbb{R} \tag{1.10}$$

as we will now show using the principle of mathematical induction: Obviously (1.10) holds for $k = 0$. Assume that (1.10) holds with k replaced by some $m \in \mathbb{N}_0$. Then

$$\begin{aligned} g_{m+1}(t,s) &= h_{m+1}(t,s) \\ &= \int_s^t \frac{(\tau - s)^m}{m!} d\tau \\ &= \left.\frac{(\tau - s)^{m+1}}{(m+1)!}\right|_{\tau=s}^{\tau=t} \\ &= \frac{(t-s)^{m+1}}{(m+1)!}, \end{aligned}$$

i.e., (1.10) holds with k replaced by $m+1$. We note that, for an n-times differentiable function $f : \mathbb{R} \to \mathbb{R}$, the following well-known Taylor's formula holds: Let $\alpha \in \mathbb{R}$ be arbitrary. Then, for all $t \in \mathbb{R}$, the representations

$$f(t) = \sum_{k=0}^{n-1} \frac{(t-\alpha)^k}{k!} f^{(k)}(\alpha) + \frac{1}{(n-1)!} \int_\alpha^t (t-\tau)^{n-1} f^{(n)}(\tau)d\tau$$

$$= \sum_{k=0}^{n-1} h_k(t,\alpha) f^{(k)}(\alpha) + \int_\alpha^t h_{n-1}(t, \sigma(\tau)) f^{(n)}(\tau)d\tau \tag{1.11}$$

$$= \sum_{k=0}^{n-1} (-1)^k g_k(\alpha, t) f^{(k)}(\alpha) + \int_\alpha^t (-1)^{n-1} g_{n-1}(\sigma(\tau), t) f^{(n)}(\tau)d\tau \tag{1.12}$$

are valid, where $f^{(k)}$ denotes as usual the kth derivative of f. Above we have used the relationship

$$(-1)^k g_k(s,t) = (-1)^k \frac{(s-t)^k}{k!} = \frac{(t-s)^k}{k!} = h_k(t,s), \tag{1.13}$$

which holds for all $k \in \mathbb{N}_0$.

Next, consider $\mathbb{T} = \mathbb{Z}$. Then $\sigma(t) = t + 1$ for $t \in \mathbb{Z}$. We have for $s, t \in \mathbb{Z}$

$$\begin{aligned} h_2(t,s) &= \int_s^t h_1(\tau,s)\Delta s = \int_s^t (\tau - s)^{(1)}\Delta s \\ &= \left[\frac{(\tau - s)^{(2)}}{2}\right]_s^t \\ &= \frac{(t-s)^{(2)}}{2!} = \binom{t-s}{2}. \end{aligned}$$

We claim that for $k \in \mathbb{N}_0$, we have

$$h_k(t,s) = \frac{(t-s)^{(k)}}{k!} = \binom{t-s}{k} \quad \text{for all} \quad s,t \in \mathbb{Z}. \tag{1.14}$$

Assume (1.14) holds for k replaced by m. Then

$$\begin{aligned} h_{m+1}(t,s) &= \int_s^t h_m(\tau,s)\Delta\tau \\ &= \int_s^t \frac{(\tau-s)^{(m)}}{m!}\Delta\tau \\ &= \frac{(t-s)^{(m+1)}}{(m+1)!}, \end{aligned}$$

which is (1.14) with k replaced by $m+1$. Hence by mathematical induction we get that (1.14) holds for all $k \in \mathbb{N}_0$. Similarly it is possible to show that

$$g_k(t,s) = \frac{(t-s+k-1)^{(k)}}{k!} \quad \text{for all} \quad s,t \in \mathbb{Z} \tag{1.15}$$

holds for all $k \in \mathbb{N}_0$. As before we observe that the relationship

$$\begin{aligned} (-1)^k g_k(s,t) &= (-1)^k \frac{(s-t+k-1)^{(k)}}{k!} \\ &= (-1)^k \frac{(s-t+k-1)(s-t+k-2)\cdots(s-t)}{k!} \\ &= \frac{(t-s)\cdots(t-s+2-k)(t-s+1-k)}{k!} \\ &= \frac{(t-s)^{(k)}}{k!} \\ &= h_k(t,s) \end{aligned} \tag{1.16}$$

holds for all $k \in \mathbb{N}_0$. The well-known discrete version of Taylor's formula (see e.g. [**5**]) reads as follows: Let $f : \mathbb{Z} \to \mathbb{R}$ be a function, and let $\alpha \in \mathbb{Z}$. Then, for all $t \in \mathbb{Z}$ with $t > \alpha + n$, the representations

$$\begin{aligned} f(t) &= \sum_{k=0}^{n-1} \frac{(t-\alpha)^{(k)}}{k!}\Delta^k f(\alpha) + \frac{1}{(n-1)!}\sum_{\tau=\alpha}^{t-n}(t-\tau-1)^{(n-1)}\Delta^n f(\tau) \\ &= \sum_{k=0}^{n-1} h_k(t,\alpha)\Delta^k f(\alpha) + \sum_{\tau=\alpha}^{t-n} h_{n-1}(t,\sigma(\tau))\Delta^n f(\tau) \qquad (1.17) \\ &= \sum_{k=0}^{n-1}(-1)^k g_k(\alpha,t)\Delta^k f(\alpha) + \sum_{\tau=\alpha}^{t-n}(-1)^{n-1}g_{n-1}(\sigma(\tau),t)\Delta^n f(\tau) \qquad (1.18) \end{aligned}$$

hold, where Δ^k is the usual k-times iterated forward difference operator.

Exercise 1.103. Verify that formula (1.15) holds for all $k \in \mathbb{N}_0$.

Example 1.104. We consider the time scale

$$\mathbb{T} = \overline{q^{\mathbb{Z}}} \quad \text{for some} \quad q > 1$$

from Example 1.41. The claim is that

$$h_k(t,s) = \prod_{\nu=0}^{k-1} \frac{t - q^\nu s}{\sum_{\mu=0}^{\nu} q^\mu} \quad \text{for all} \quad s,t \in \mathbb{T} \tag{1.19}$$

holds for all $k \in \mathbb{N}_0$. Obviously, for $k = 0$ (observe that the empty product is considered to be 1, as usual), the claim (1.19) holds. Now we assume (1.19) holds with k replaced by some $m \in \mathbb{N}_0$. Then

$$\begin{aligned}
\left\{\prod_{\nu=0}^{m} \frac{t - q^\nu s}{\sum_{\mu=0}^{\nu} q^\mu}\right\}^\Delta &= \frac{\prod_{\nu=0}^{m} \frac{\sigma(t) - q^\nu s}{\sum_{\mu=0}^{\nu} q^\mu} - \prod_{\nu=0}^{m} \frac{t - q^\nu s}{\sum_{\mu=0}^{\nu} q^\mu}}{\mu(t)} \\
&= \frac{\frac{qt - q^m s}{\sum_{\mu=0}^{m} q^\mu} h_m(\sigma(t),s) - \frac{t - q^m s}{\sum_{\mu=0}^{m} q^\mu} h_m(t,s)}{\mu(t)} \\
&= \frac{qt - q^m s}{\mu(t) \sum_{\mu=0}^{m} q^\mu} \left\{h_m(t,s) + \mu(t) h_m^\Delta(t,s)\right\} - \frac{t - q^m s}{\mu(t) \sum_{\mu=0}^{m} q^\mu} h_m(t,s) \\
&= \frac{qt - t}{\mu(t) \sum_{\mu=0}^{m} q^\mu} h_m(t,s) + \frac{qt - q^m s}{\sum_{\mu=0}^{m} q^\mu} h_m^\Delta(t,s) \\
&= \frac{1}{\sum_{\mu=0}^{m} q^\mu} h_m(t,s) + \frac{qt - q^m s}{\sum_{\mu=0}^{m} q^\mu} h_{m-1}(t,s) \\
&= \frac{1}{\sum_{\mu=0}^{m} q^\mu} \left\{h_m(t,s) + (qt - q^m s) \prod_{\nu=0}^{m-2} \frac{t - q^\nu s}{\sum_{\mu=0}^{\nu} q^\mu}\right\} \\
&= \frac{1}{\sum_{\mu=0}^{m} q^\mu} \left\{h_m(t,s) + q \left(\sum_{\mu=0}^{m-1} q^\mu\right) \prod_{\nu=0}^{m-1} \frac{t - q^\nu s}{\sum_{\mu=0}^{\nu} q^\mu}\right\} \\
&= \frac{h_m(t,s)}{\sum_{\mu=0}^{m} q^\mu} \left\{1 + q \sum_{\mu=0}^{m-1} q^\mu\right\} \\
&= h_m(t,s) \frac{1 + \sum_{\mu=1}^{m} q^\mu}{\sum_{\mu=0}^{m} q^\mu} \\
&= h_m(t,s)
\end{aligned}$$

so that (1.19) follows with k replaced by $m+1$. Hence, by the principle of mathematical induction, (1.19) holds for all $k \in \mathbb{N}_0$.

As a special case, we consider the choice $q = 2$. This yields

$$h_k(t,s) = \prod_{\nu=0}^{k-1} \frac{t - 2^\nu s}{2^{\nu+1} - 1} \quad \text{for all} \quad s,t \in \mathbb{T}.$$

E.g., we have

$$h_2(t,s) = \frac{(t-s)(t-2s)}{3}, \quad h_3(t,s) = \frac{(t-s)(t-2s)(t-4s)}{21},$$

and

$$h_4(t,s) = \frac{(t-s)(t-2s)(t-4s)(t-8s)}{315}.$$

Exercise 1.105. Find the functions g_k for $k \in \mathbb{N}_0$, where $\mathbb{T}$ is the time scale considered in Example 1.104.

Exercise 1.106. Let $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ for some $q > 1$. For $n \in \mathbb{N}$, evaluate

$$\int_0^t s^n \Delta s.$$

Exercise 1.107. Find the functions $h_k(\cdot, 0)$, for $k = 0, 1, 2, 3$ if $\mathbb{T} := [0,1] \cup [3,4]$.

Now we will present and prove Taylor's formula for the case of a general time scale $\mathbb{T}$. First we need three preliminary results.

Lemma 1.108. *Let $n \in \mathbb{N}$. Suppose f is n-times differentiable and p_k, $0 \le k \le n-1$, are differentiable at some $t \in \mathbb{T}$ with*

$$p_{k+1}^{\Delta}(t) = p_k^{\sigma}(t) \quad \textit{for all} \quad 0 \le k \le n-2. \tag{1.20}$$

Then we have at t

$$\left[\sum_{k=0}^{n-1} (-1)^k f^{\Delta^k} p_k\right]^{\Delta} = (-1)^{n-1} f^{\Delta^n} p_{n-1}^{\sigma} + f p_0^{\Delta}.$$

Proof. Using Theorem 1.20 (i), (ii), (iii), and (1.20) we find that

$$\begin{aligned}
\left[\sum_{k=0}^{n-1} (-1)^k f^{\Delta^k} p_k\right]^{\Delta} &= \sum_{k=0}^{n-1} (-1)^k \left[f^{\Delta^k} p_k\right]^{\Delta} \\
&= \sum_{k=0}^{n-1} (-1)^k \left[f^{\Delta^{k+1}} p_k^{\sigma} + f^{\Delta^k} p_k^{\Delta}\right] \\
&= \sum_{k=0}^{n-2} (-1)^k f^{\Delta^{k+1}} p_k^{\sigma} + (-1)^{n-1} f^{\Delta^n} p_{n-1}^{\sigma} + \sum_{k=1}^{n-1} (-1)^k f^{\Delta^k} p_k^{\Delta} + f p_0^{\Delta} \\
&= \sum_{k=0}^{n-2} (-1)^k f^{\Delta^{k+1}} p_k^{\sigma} + (-1)^{n-1} f^{\Delta^n} p_{n-1}^{\sigma} - \sum_{k=0}^{n-2} (-1)^k f^{\Delta^{k+1}} p_{k+1}^{\Delta} + f p_0^{\Delta} \\
&= (-1)^{n-1} f^{\Delta^n} p_{n-1}^{\sigma} + f p_0^{\Delta}
\end{aligned}$$

holds at t. This proves the lemma. □

Lemma 1.109. *The functions g_k defined in* (1.7) *and* (1.8) *satisfy for all $t \in \mathbb{T}$*

$$g_n(\rho^k(t), t) = 0 \quad \textit{for all} \quad n \in \mathbb{N} \quad \textit{and all} \quad 0 \le k \le n-1.$$

Proof. We prove this result by induction. First, for $k = 0$, we have

$$g_n(\rho^0(t), t) = g_n(t,t) = 0.$$

To complete the induction it suffices to show that

$$g_{n-1}(\rho^k(t), t) = g_n(\rho^k(t), t) = 0 \quad \text{with} \quad 0 \le k < n$$

implies that

$$g_n(\rho^{k+1}(t), t) = 0.$$

First, if $\rho^k(t)$ is left-dense, then $\rho^{k+1}(t) = \rho^k(t)$ so that

$$g_n(\rho^{k+1}(t), t) = g_n(\rho^k(t), t) = 0.$$

If $\rho^k(t)$ is not left-dense, then it is left-scattered, and $\sigma(\rho^{k+1}(t)) = \rho^k(t)$ so that by Theorem 1.16 (iv) and (1.8)

$$\begin{aligned} g_n(\rho^{k+1}(t), t) &= g_n(\sigma(\rho^{k+1}(t)), t) - \mu(\rho^{k+1}(t)) g_n^{\Delta}(\rho^{k+1}(t), t) \\ &= g_n(\rho^k(t), t) - \mu(\rho^{k+1}(t)) g_{n-1}(\sigma(\rho^{k+1}(t)), t) \\ &= g_n(\rho^k(t), t) - \mu(\rho^{k+1}(t)) g_{n-1}(\rho^k(t), t) \\ &= 0 \end{aligned}$$

(observe $n \neq 1$). This proves our claim. ☐

Lemma 1.110. *Let $n \in \mathbb{N}$, $t \in \mathbb{T}$, and suppose that f is $(n-1)$-times differentiable at $\rho^{n-1}(t)$. Then we have*

$$\sum_{k=0}^{n-1} (-1)^k f^{\Delta^k}(\rho^{n-1}(t)) g_k(\rho^{n-1}(t), t) = f(t), \tag{1.21}$$

where the functions g_k are defined by (1.7) and (1.8).

Proof. First of all we have

$$(-1)^0 f^{\Delta^0}(\rho^0(t)) g_0(\rho^0(t), t) = f(t) g_0(t, t) = f(t)$$

so that (1.21) is true for $n = 1$. Suppose now that (1.21) holds with n replaced by some $m \in \mathbb{N}$. Then we consider two cases. First, if $\rho^{m-1}(t)$ is left-dense, then $\rho^m(t) = \rho^{m-1}(t)$ and hence

$$\begin{aligned} \sum_{k=0}^{m} (-1)^k f^{\Delta^k}(\rho^m(t)) g_k(\rho^m(t), t) &= \sum_{k=0}^{m-1} (-1)^k f^{\Delta^k}(\rho^m(t)) g_k(\rho^m(t), t) \\ &\quad + (-1)^m f^{\Delta^m}(\rho^m(t)) g_m(\rho^m(t), t) \\ &= \sum_{k=0}^{m-1} (-1)^k f^{\Delta^k}(\rho^{m-1}(t)) g_k(\rho^{m-1}(t), t) + (-1)^m f^{\Delta^m}(\rho^{m-1}(t)) g_m(\rho^{m-1}(t), t) \\ &= f(t) + (-1)^m f^{\Delta^m}(\rho^{m-1}(t)) g_m(\rho^{m-1}(t), t) \\ &= f(t), \end{aligned}$$

where we used Lemma 1.109 to obtain the last equation. Hence (1.21) holds with n replaced by $m+1$. We have to draw the same conclusion for the case that $\rho^m(t)$ is left-scattered. In this case we have $\sigma(\rho^m(t)) = \rho^{m-1}(t)$ and hence by Theorem 1.16 (iv) and (1.8) for $k \in \mathbb{N}$

$$\begin{aligned} g_k(\rho^{m-1}(t), t) &= g_k(\sigma(\rho^m(t)), t) \\ &= g_k(\rho^m(t), t) + \mu(\rho^m(t)) g_k^{\Delta}(\rho^m(t), t) \\ &= g_k(\rho^m(t), t) + \mu(\rho^m(t)) g_{k-1}(\sigma(\rho^m(t)), t) \\ &= g_k(\rho^m(t), t) + \mu(\rho^m(t)) g_{k-1}(\rho^{m-1}(t), t). \end{aligned}$$

Therefore we conclude (apply Lemma 1.109 for the third equation)

$$\begin{aligned}
&\sum_{k=0}^{m}(-1)^k f^{\Delta^k}(\rho^m(t))g_k(\rho^m(t),t)\\
&= f(\rho^m(t)) + \sum_{k=1}^{m}(-1)^k f^{\Delta^k}(\rho^m(t))g_k(\rho^m(t),t)\\
&= f(\rho^m(t))\\
&\quad + \sum_{k=1}^{m}(-1)^k f^{\Delta^k}(\rho^m(t))\left[g_k(\rho^{m-1}(t),t) - \mu(\rho^m(t))g_{k-1}(\rho^{m-1}(t),t)\right]\\
&= f(\rho^m(t)) + \sum_{k=1}^{m-1}(-1)^k f^{\Delta^k}(\rho^m(t))g_k(\rho^{m-1}(t),t)\\
&\quad + \sum_{k=1}^{m}(-1)^{k-1}\mu(\rho^m(t))f^{\Delta^k}(\rho^m(t))g_{k-1}(\rho^{m-1}(t),t)\\
&= \sum_{k=0}^{m-1}(-1)^k f^{\Delta^k}(\rho^m(t))g_k(\rho^{m-1}(t),t)\\
&\quad + \sum_{k=0}^{m-1}(-1)^k\mu(\rho^m(t))f^{\Delta^{k+1}}(\rho^m(t))g_k(\rho^{m-1}(t),t)\\
&= \sum_{k=0}^{m-1}(-1)^k\left[f^{\Delta^k}(\rho^m(t)) + \mu(\rho^m(t))(f^{\Delta^k})^{\Delta}(\rho^m(t))\right]g_k(\rho^{m-1}(t),t)\\
&= \sum_{k=0}^{m-1}(-1)^k (f^{\Delta^k})^{\sigma}(\rho^m(t))g_k(\rho^{m-1}(t),t)\\
&= \sum_{k=0}^{m-1}(-1)^k (f^{\Delta^k})(\sigma(\rho^m(t)))g_k(\rho^{m-1}(t),t)\\
&= \sum_{k=0}^{m-1}(-1)^k (f^{\Delta^k})(\rho^{m-1}(t))g_k(\rho^{m-1}(t),t)\\
&= f(t).
\end{aligned}$$

As before, (1.21) holds with n replaced by $m+1$, and an application of the principle of mathematical induction finishes the proof. ☐

Theorem 1.111 (Taylor's Formula). *Let $n \in \mathbb{N}$. Suppose f is n-times differentiable on $\mathbb{T}^{\kappa^n}$. Let $\alpha \in \mathbb{T}^{\kappa^{n-1}}$, $t \in \mathbb{T}$, and define the functions g_k by* (1.7) *and* (1.8), *i.e.,*

$$g_0(r,s) \equiv 1 \quad \textit{and} \quad g_{k+1}(r,s) = \int_s^r g_k(\sigma(\tau),s)\Delta\tau \ \textit{for}\ k \in \mathbb{N}_0.$$

Then we have

$$f(t) = \sum_{k=0}^{n-1}(-1)^k g_k(\alpha,t)f^{\Delta^k}(\alpha) + \int_{\alpha}^{\rho^{n-1}(t)}(-1)^{n-1}g_{n-1}(\sigma(\tau),t)f^{\Delta^n}(\tau)\Delta\tau.$$

Proof. By Lemma 1.108 we have

$$\left[\sum_{k=0}^{n-1}(-1)^k g_k(\cdot,t)f^{\Delta^k}\right]^{\Delta}(\tau)=(-1)^{n-1}g_{n-1}(\sigma(\tau),t)f^{\Delta^n}(\tau)$$

for all $\tau\in\mathbb{T}^{\kappa^n}$. Since $\alpha,\rho^{n-1}(t)\in\mathbb{T}^{\kappa^{n-1}}$, we may integrate the above equation from α to $\rho^{n-1}(t)$ to obtain

$$\begin{aligned}\int_\alpha^{\rho^{n-1}(t)}(-1)^{n-1}g_{n-1}(\sigma(\tau),t)f^{\Delta^n}(\tau)\Delta\tau &= \int_\alpha^{\rho^{n-1}(t)}\left[\sum_{k=0}^{n-1}(-1)^k g_k(\cdot,t)f^{\Delta^k}\right]^{\Delta}(\tau)\Delta\tau\\ &= \sum_{k=0}^{n-1}(-1)^k g_k(\rho^{n-1}(t),t)f^{\Delta^k}(\rho^{n-1}(t))-\sum_{k=0}^{n-1}(-1)^k g_k(\alpha,t)f^{\Delta^k}(\alpha)\\ &= f(t)-\sum_{k=0}^{n-1}(-1)^k g_k(\alpha,t)f^{\Delta^k}(\alpha),\end{aligned}$$

where we used formula (1.21) from Lemma 1.110. □

Our first application of Theorem 1.111 yields an alternative form of Taylor's formula in terms of the functions h_k rather than g_k.

Theorem 1.112. *The functions g_k and h_k defined in* (1.7), (1.8), *and* (1.9) *satisfy*

$$h_n(t,s)=(-1)^n g_n(s,t)\quad \textit{for all}\quad t\in\mathbb{T}\quad \textit{and all}\quad s\in\mathbb{T}^{\kappa^n}.$$

Proof. We let $t\in\mathbb{T}$, $s\in\mathbb{T}^{\kappa^n}$, and apply Theorem 1.111 with

$$\alpha=s\quad\text{and}\quad f=h_n(\cdot,s).$$

This yields $f^\Delta=h_{n-1}(\cdot,s)$ and (apply (1.9) successively)

$$f^{\Delta^k}=h_{n-k}(\cdot,s)\quad\text{for all}\quad 0\le k\le n.$$

Therefore

$$f^{\Delta^k}(s)=h_{n-k}(s,s)\quad\text{for all}\quad 0\le k\le n-1,$$

$f^{\Delta^n}(s)=h_0(s,s)=1$, and $f^{\Delta^{n+1}}(\tau)\equiv 0$. An application of Theorem 1.111 now shows

$$\begin{aligned}h_n(t,s) &= f(t)\\ &= \sum_{k=0}^{n}(-1)^k g_k(\alpha,t)f^{\Delta^k}(\alpha)+\int_\alpha^{\rho^n(t)}(-1)^n g_n(\sigma(\tau),t)f^{\Delta^{n+1}}(\tau)\Delta\tau\\ &= \sum_{k=0}^{n}(-1)^k g_k(s,t)f^{\Delta^k}(s)+\int_s^{\rho^n(t)}(-1)^n g_n(\sigma(\tau),t)f^{\Delta^{n+1}}(\tau)\Delta\tau\\ &= (-1)^n g_n(s,t)f^{\Delta^n}(s)\\ &= (-1)^n g_n(s,t),\end{aligned}$$

and this completes the proof. □

Theorem 1.113 (Taylor's Formula). *Let $n \in \mathbb{N}$. Suppose f is n-times differentiable on $\mathbb{T}^{\kappa^n}$. Let $\alpha \in \mathbb{T}^{\kappa^{n-1}}$, $t \in \mathbb{T}$, and define the functions h_k by (1.7) and (1.9), i.e.,*

$$h_0(r,s) \equiv 1 \quad \text{and} \quad h_{k+1}(r,s) = \int_s^r h_k(\tau,s)\Delta\tau \text{ for } k \in \mathbb{N}_0.$$

Then we have

$$f(t) = \sum_{k=0}^{n-1} h_k(t,\alpha) f^{\Delta^k}(\alpha) + \int_\alpha^{\rho^{n-1}(t)} h_{n-1}(t,\sigma(\tau)) f^{\Delta^n}(\tau)\Delta\tau.$$

Proof. This is a direct consequence of Theorem 1.111 and Theorem 1.112. □

Remark 1.114. The reader may compare Example 1.102 (i.e., the cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$) to the above presented theory. Theorem 1.112 is reflected in formulas (1.10) and (1.16). While the first version of Taylor's formula, Theorem 1.111, corresponds to formulas (1.12) and (1.18), the second version, Theorem 1.113, corresponds to formulas (1.11) and (1.17).

1.7. Further Basic Results

We now state the intermediate value theorem for a continuous function on a time scale.

Theorem 1.115 (Intermediate Value Theorem). *Assume $x : \mathbb{T} \to \mathbb{R}$ is continuous, $a < b$ are points in $\mathbb{T}$, and*

$$x(a)x(b) < 0.$$

Then there exists $c \in [a,b)$ such that either $x(c) = 0$ or

$$x(c)x^\sigma(c) < 0.$$

Exercise 1.116. Prove the above intermediate value theorem.

We will use the following result later. By $f^\Delta(t,\tau)$ in the following theorem we mean for each fixed τ the delta derivative of $f(t,\tau)$ with respect to t.

Theorem 1.117. *Let $a \in \mathbb{T}^\kappa$, $b \in \mathbb{T}$ and assume $f : \mathbb{T} \times \mathbb{T}^\kappa \to \mathbb{R}$ is continuous at (t,t), where $t \in \mathbb{T}^\kappa$ with $t > a$. Suppose that for all $\varepsilon > 0$ there exists a neighborhood U of t (independent of τ) such that*

$$|f(\sigma(t),\tau) - f(s,\tau) - f^\Delta(t,\tau)(\sigma(t)-s)| \le \varepsilon|\sigma(t)-s| \quad \text{for all} \quad s \in U,$$

where f^Δ denotes the derivative of f with respect to the first variable. Then

(i) $g(t) := \int_a^t f(t,\tau)\Delta\tau$ *implies* $g^\Delta(t) = \int_a^t f^\Delta(t,\tau)\Delta\tau + f(\sigma(t),t)$;
(ii) $h(t) := \int_t^b f(t,\tau)\Delta\tau$ *implies* $h^\Delta(t) = \int_t^b f^\Delta(t,\tau)\Delta\tau - f(\sigma(t),t)$.

Proof. We only prove (i) while the proof of (ii) is similar and is left to the reader in Exercise 1.118. Let $\varepsilon > 0$. By assumption there exists a neighborhood U_1 of t such that

$$|f(\sigma(t),\tau) - f(s,\tau) - f^\Delta(t,\tau)(\sigma(t)-s)| \le \frac{\varepsilon}{2(\sigma(t)-a)}|\sigma(t)-s| \quad \text{for all} \quad s \in U_1.$$

Since f is continuous at (t,t), there exists a neighborhood U_2 of t such that

$$|f(s,\tau) - f(t,t)| \le \frac{\varepsilon}{2} \quad \text{whenever} \quad s,\tau \in U_2.$$

Now define $U = U_1 \cap U_2$ and let $s \in U$. Then

$$
\begin{aligned}
&\left| g(\sigma(t)) - g(s) - \left[f(\sigma(t), t) + \int_a^t f^{\Delta}(t,\tau)\Delta\tau \right] (\sigma(t) - s) \right| \\
= & \left| \int_a^{\sigma(t)} f(\sigma(t),\tau)\Delta\tau - \int_a^s f(s,\tau)\Delta\tau - (\sigma(t)-s) f(\sigma(t),t) \right. \\
& \left. -(\sigma(t)-s)\int_a^t f^{\Delta}(t,\tau)\Delta\tau \right| \\
= & \left| \int_a^{\sigma(t)} \left[f(\sigma(t),\tau) - f(s,\tau) - f^{\Delta}(t,\tau)(\sigma(t)-s) \right] \Delta\tau \right. \\
& \left. - \int_{\sigma(t)}^s f(s,\tau)\Delta\tau - (\sigma(t)-s) f(\sigma(t),t) - (\sigma(t)-s)\int_{\sigma(t)}^t f^{\Delta}(t,\tau)\Delta\tau \right| \\
= & \left| \int_a^{\sigma(t)} \left[f(\sigma(t),\tau) - f(s,\tau) - f^{\Delta}(t,\tau)(\sigma(t)-s) \right] \Delta\tau \right. \\
& \left. - \int_{\sigma(t)}^s f(s,\tau)\Delta\tau - (\sigma(t)-s) f(\sigma(t),t) + (\sigma(t)-s)\mu(t) f^{\Delta}(t,t) \right| \\
= & \left| \int_a^{\sigma(t)} \left[f(\sigma(t),\tau) - f(s,\tau) - f^{\Delta}(t,\tau)(\sigma(t)-s) \right] \Delta\tau \right. \\
& \left. + \int_s^{\sigma(t)} f(s,\tau)\Delta\tau - (\sigma(t)-s) f(t,t) \right| \\
= & \left| \int_a^{\sigma(t)} \left[f(\sigma(t),\tau) - f(s,\tau) - f^{\Delta}(t,\tau)(\sigma(t)-s) \right] \Delta\tau \right. \\
& \left. + \int_s^{\sigma(t)} [f(s,\tau) - f(t,t)]\Delta\tau \right| \\
\leq & \int_a^{\sigma(t)} \left| f(\sigma(t),\tau) - f(s,\tau) - f^{\Delta}(t,\tau)(\sigma(t)-s) \right| \Delta\tau \\
& + \int_s^{\sigma(t)} |f(s,\tau) - f(t,t)| \Delta\tau \\
\leq & \int_a^{\sigma(t)} \frac{\varepsilon}{2(\sigma(t)-a)} |\sigma(t)-s| \Delta\tau + \left| \int_s^{\sigma(t)} \frac{\varepsilon}{2}\Delta\tau \right| \\
= & \frac{\varepsilon}{2}|\sigma(t)-s| + \frac{\varepsilon}{2}|\sigma(t)-s| \\
= & \varepsilon|\sigma(t)-s|,
\end{aligned}
$$

where we also have used Theorem 1.75 and Theorem 1.16 (iv). ☐

Exercise 1.118. Prove Theorem 1.117 (ii).

Finally, we present several versions of L'Hôspital's rule. We let

$$\overline{\mathbb{T}} = \mathbb{T} \cup \{\sup \mathbb{T}\} \cup \{\inf \mathbb{T}\}.$$

If $\infty \in \overline{\mathbb{T}}$, we call ∞ left-dense, and $-\infty$ is called right-dense provided $-\infty \in \overline{\mathbb{T}}$. For any left-dense $t_0 \in \mathbb{T}$ and any $\varepsilon > 0$, the set

$$L_\varepsilon(t_0) = \{t \in \mathbb{T}: \ 0 < t_0 - t < \varepsilon\}$$

is nonempty, and so is $L_\varepsilon(\infty) = \left\{t \in \mathbb{T}: \ t > \frac{1}{\varepsilon}\right\}$ if $\infty \in \overline{\mathbb{T}}$. The sets $R_\varepsilon(t_0)$ for right-dense $t_0 \in \overline{\mathbb{T}}$ and $\varepsilon > 0$ are defined accordingly. For a function $h : \mathbb{T} \to \mathbb{R}$ we define

$$\liminf_{t \to t_0^-} h(t) = \lim_{\varepsilon \to 0^+} \inf_{t \in L_\varepsilon(t_0)} h(t) \quad \text{for left-dense} \quad t_0 \in \overline{\mathbb{T}},$$

and $\liminf_{t \to t_0^+} h(t)$, $\limsup_{t \to t_0^-} h(t)$, $\limsup_{t \to t_0^+} h(t)$ are defined analogously.

Theorem 1.119 (L'Hôspital's Rule)**.** *Assume f and g are differentiable on $\mathbb{T}$ with*

$$\lim_{t \to t_0^-} f(t) = \lim_{t \to t_0^-} g(t) = 0 \quad \textit{for some left-dense} \quad t_0 \in \overline{\mathbb{T}}. \tag{1.22}$$

Suppose there exists $\varepsilon > 0$ with

$$g(t) > 0, \ g^\Delta(t) < 0 \quad \textit{for all} \quad t \in L_\varepsilon(t_0). \tag{1.23}$$

Then we have

$$\liminf_{t \to t_0^-} \frac{f^\Delta(t)}{g^\Delta(t)} \le \liminf_{t \to t_0^-} \frac{f(t)}{g(t)} \le \limsup_{t \to t_0^-} \frac{f(t)}{g(t)} \le \limsup_{t \to t_0^-} \frac{f^\Delta(t)}{g^\Delta(t)}.$$

Proof. Let $\delta \in (0, \varepsilon]$ and put $\alpha = \inf_{\tau \in L_\delta(t_0)} \frac{f^\Delta(\tau)}{g^\Delta(\tau)}$, $\beta = \sup_{\tau \in L_\delta(t_0)} \frac{f^\Delta(\tau)}{g^\Delta(\tau)}$. Then

$$\alpha g^\Delta(\tau) \ge f^\Delta(\tau) \ge \beta g^\Delta(\tau) \quad \text{for all} \quad \tau \in L_\delta(t_0)$$

by (1.23) and hence

$$\int_s^t \alpha g^\Delta(\tau)\Delta\tau \ge \int_s^t f^\Delta(\tau)\Delta\tau \ge \int_s^t \beta g^\Delta(\tau)\Delta\tau \text{ for all } s, t \in L_\delta(t_0), \ s < t$$

so that

$$\alpha g(t) - \alpha g(s) \ge f(t) - f(s) \ge \beta g(t) - \beta g(s) \text{ for all } s, t \in L_\delta(t_0), \ s < t.$$

Now, letting $t \to t_0^-$, we find from (1.22)

$$-\alpha g(s) \ge -f(s) \ge -\beta g(s) \quad \text{for all} \quad s \in L_\delta(t_0)$$

and hence by (1.23)

$$\inf_{\tau \in L_\delta(t_0)} \frac{f^\Delta(\tau)}{g^\Delta(\tau)} = \alpha \le \inf_{s \in L_\delta(t_0)} \frac{f(s)}{g(s)} \le \sup_{s \in L_\delta(t_0)} \frac{f(s)}{g(s)} \le \beta = \sup_{\tau \in L_\delta(t_0)} \frac{f^\Delta(\tau)}{g^\Delta(\tau)}.$$

Letting $\delta \to 0^+$ yields our desired result. □

Theorem 1.120 (L'Hôspital's Rule)**.** *Assume f and g are differentiable on $\mathbb{T}$ with*

$$\lim_{t \to t_0^-} g(t) = \infty \quad \textit{for some left-dense} \quad t_0 \in \overline{\mathbb{T}}. \tag{1.24}$$

Suppose there exists $\varepsilon > 0$ with

$$g(t) > 0, \ g^\Delta(t) > 0 \quad \textit{for all} \quad t \in L_\varepsilon(t_0). \tag{1.25}$$

Then $\lim_{t \to t_0^-} \frac{f^\Delta(t)}{g^\Delta(t)} = r \in \overline{\mathbb{R}}$ implies $\lim_{t \to t_0^-} \frac{f(t)}{g(t)} = r$.

Proof. First suppose $r \in \mathbb{R}$. Let $c > 0$. Then there exists $\delta \in (0, \varepsilon]$ such that

$$\left| \frac{f^\Delta(\tau)}{g^\Delta(\tau)} - r \right| \le c \quad \text{for all} \quad \tau \in L_\delta(t_0)$$

and hence by (1.25)

$$-cg^\Delta(\tau) \le f^\Delta(\tau) - rg^\Delta(\tau) \le cg^\Delta(\tau) \quad \text{for all} \quad \tau \in L_\delta(t_0).$$

We integrate as in the proof of Theorem 1.119 and use (1.25) to obtain

$$(r-c)\left(1 - \frac{g(s)}{g(t)}\right) \le \frac{f(t)}{g(t)} - \frac{f(s)}{g(t)} \le (r+c)\left(1 - \frac{g(s)}{g(t)}\right) \text{ for all } s, t \in L_\delta(t_0); s < t.$$

Letting $t \to t_0^-$ and applying (1.24) yields

$$r - c \le \liminf_{t \to t_0^-} \frac{f(t)}{g(t)} \le \limsup_{t \to t_0^-} \frac{f(t)}{g(t)} \le r + c.$$

Now we let $c \to 0^+$ to see that $\lim_{t \to t_0^-} \frac{f(t)}{g(t)}$ exists and equals r.

Next, if $r = \infty$ (and similarly if $r = -\infty$), let $c > 0$. Then there exists $\delta \in (0, \varepsilon]$ with

$$\frac{f^\Delta(\tau)}{g^\Delta(\tau)} \ge \frac{1}{c} \quad \text{for all} \quad \tau \in L_\delta(t_0)$$

and hence by (1.25)

$$f^\Delta(\tau) \ge \frac{1}{c} g^\Delta(\tau) \quad \text{for all} \quad \tau \in L_\delta(t_0).$$

We integrate again to get

$$\frac{f(t)}{g(t)} - \frac{f(s)}{g(t)} \ge \frac{1}{c}\left(1 - \frac{g(s)}{g(t)}\right) \quad \text{for all} \quad s, t \in L_\delta(t_0); s < t.$$

Thus, letting $t \to t_0^-$ and applying (1.24), we find $\liminf_{t \to t_0^-} \frac{f(t)}{g(t)} \ge \frac{1}{c}$, and then, letting $c \to 0^+$, we obtain $\lim_{t \to t_0^-} \frac{f(t)}{g(t)} = \infty = r$. □

1.8. Notes and References

The calculus of measure chains originated in 1988, when Stefan Hilger completed his PhD thesis [**159**] "Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten" at Universität Würzburg, Germany, under the supervision of Bernd Aulbach. The first publications on the subject of measure chains are Hilger [**160**] and Aulbach and Hilger [**49, 50**].

The basic definitions of jump operators and delta differentiation are due to Stefan Hilger. We have included some previously unpublished but easy examples in the section on differentiation, e.g., the derivatives of t^k for $k \in \mathbb{Z}$ in Theorem 1.24 and the Leibniz formula for the nth derivative of a product of two functions in Theorem 1.32. This first section also contains the induction principle on time scales. It is essentially contained in Dieudonné [**110**]. For the proofs of some of the main existence results, this induction principle is utilized, while the remaining results presented in this book can be derived without using this induction principle. The induction principle requires distinction of several cases, depending on whether the point under consideration is left-dense, left-scattered, right-dense, or right-scattered. However, it is one of the key features of the time scales calculus to unify proofs for the continuous and the discrete cases, and hence it should, wherever possible, be avoided to start discussing those several cases independently. For that purpose it is helpful to have formulas available that are valid at each element of the time scale, as e.g. in Theorem 1.16 (iv) or the product and quotient rules in Theorem 1.20.

Section 1.3 contains many examples that are considered throughout this book,. Concerning the two main examples of the time scales calculus, we refer to the classical books [**103, 152, 237**] for differential equations and [**5, 125, 191, 200, 216**] for difference equations. Other examples discussed in this section contain the integer multiples of a number $h > 0$, denoted by $h\mathbb{Z}$; the union of closed intervals, denoted by $\mathbb{P}$; the integer powers of a number $q > 1$, denoted by $\overline{q^{\mathbb{Z}}}$; the integer squares, denoted by $\mathbb{Z}^2$; the harmonic numbers, denoted by H_n; and the Cantor set. Many of these examples are considered in this book for the first time or are contained in [**90**]. The example on the electric circuit is taken from Stefan Keller's PhD thesis [**190**], 1999, at Universität Augsburg, Germany, under the supervision of Bernd Aulbach.

In the section on integration we define the two crucial notions of rd-continuity and regularity. These are classes of functions that possess an antiderivative and a pre-antiderivative, respectively. We present the corresponding existence theorems; however, delay most of their proofs to the last chapter in a more general setting. A rather weak form of an integral, the Cauchy integral, is defined in terms of antiderivatives. It would be of interest to derive an integral for a more general class of functions than the ones presented here, and this might be one direction of future research. Theorem 1.65 has been discussed with Roman Hilscher and Ronald Mathsen.

Section 1.5 contains several versions of the chain rule, which, in the time scales setting, does not appear in its usual form. Theorem 1.90 is due to Christian Pötzsche [**234**], and it also appears in Stefan Keller's PhD thesis [**190**]. The chain rule given in Theorem 1.93 originated in the study of so-called alpha derivatives. These alpha derivatives are introduced in Calvin Ahlbrandt, Martin Bohner, and Jerry Ridenhour [**24**], and some related results are presented in the last chapter. This topic on alpha derivatives could become a future area of research.

The time scales versions of L'Hôspital's rules given in the last section are taken from Ravi Agarwal and Martin Bohner [**9**]. Most of the results presented in the section on polynomials are contained in [**9**] as well. The functions g_k and h_k are the time scales "substitutes" for the usual polynomials. There are two of them, and that is why there are also two versions of Taylor's theorem, one using the functions g_k and the other one using the functions h_k as coefficients. The functions g_k and h_k could also be used when expressing functions defined on time scales in terms of "power" series, but this question is not addressed in this book. It also remains an open area of future research, and one could investigate how solutions of dynamic equations (see the next chapter) may be expressible in terms of such "power" series.

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