MEANS ON ARC-LIKE CONTINUA

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A *mean* on a topological space X is defined as a mapping $\mu : X \times X \to X$ such that $\mu(x,y) = \mu(y,x)$ and $\mu(x,x) = x$ for every $x, y \in X$ (in other words, it is a symmetric, idempotent, continuous binary operation on X). In [30, p. 285] an approach to this concept is presented from the standpoint of the theory of hyperspaces (a mean on a continuum X can be defined as a retraction of the hyperspace $F_2(X)$ onto $F_1(X)$, see also [21, Section 76, p. 371]; compare also [16, Section 5, p. 18] and [17, Section 6, p. 496]).

A natural problem that is related to this concept is what spaces, in particular what metric continua, admit a mean? No characterization is known till now.

It is easy to give an example of a mean on the closed unit interval $[0, 1]$ (e.g., the arithmetic mean $\mu(x,y) = \frac{x+y}{2}$). Means on $[0, 1]$, even in a more general setting, were studied by A. N. Kolmogoroff who described a structural form of these mappings in [24]. Functional equations of the type

$$(1) \qquad\qquad f(\mu(x,y)) = \mu(f(x), f(y))$$

with a given mean μ on $[0, 1]$ and unknown mapping $f : [0, 1] \to [0, 1]$ have been studied extensively, see [1]. Invertedly, a question about the existence of a mean on $[0, 1]$ for a given mapping f such that (1) holds for all $x, y \in [0, 1]$ is also discussed in some papers. E.g., in [9] it is shown that equation (1) has no solutions μ for the tent map f (see [14] for an extension) and it is asked if a surjection f on $[0, 1]$ satisfying (1) for some mean μ must necessarily be monotone.

A study on basic properties of means defined on arbitrary spaces started with the habilitation thesis of G. Aumann [2] and [3], and it was developed in [4], where it is shown that the circle, or even k-dimensional sphere for each $k \geq 1$ does not admit any mean, while each *dendrite* (i.e., a locally connected metric continuum containing no simple closed curve) does. An outline of a quite different proof

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that the circle does not admit any mean is given in [30, (0.71.1), p. 50]. These fundamental results have been generalized later in several ways.

Given a mapping $f : X \to Y$, a mapping $h : Y \to X$ is called a *right inverse of* f provided that $f \circ h = \mathrm{id}\,|Y$. If, for a given f, there exists a right inverse of f, then f is called an *r-mapping*. Each r-mapping is surjective. Let $f : X \to Y \subset X$ be a retraction (i.e. such that $f|Y = \mathrm{id}\,|Y$; then Y is called a *retract* of X). Then $h = f|Y$ is a right inverse of f, so each retraction is an r-mapping. It is known that if a space X admits a mean and $f : X \to Y$ is an r-mapping, then Y also admits a mean, [27]. In particular, each retract of X admits a mean, [33].

A continuum X is said to be *unicoherent* provided that for each decomposition of X into two subcontinua, their intersection is connected. It is known that if a locally connected metric continuum admits a mean, then it is unicoherent; if, in addition, it is 1-dimensional, then it is a dendrite, see [33] (compare also [16, Theorem 5.31, p. 22]). Local connectedness is essential in this result because the dyadic solenoid is 1-dimensional, unicoherent, and admits a mean, see [21, 76.6, p. 374] (also [16, 5.47, p. 24]; it admits an open and monotone mean, [22, Example 5]). For further progress see [6], [8], [27], [28], [29].

In an early period of studies on means, majority of results was related to locally connected spaces. One of the first examples of non-locally connected continua that admit no mean was the $\sin(1/x)$-curve, [7] (for an extension of this result see [5]). This curve is acyclic (in the sense that all its homology groups are trivial). Known examples of locally connected continua that do not admit any mean are cyclic. So, a question arises if cyclicity is the only obstruction which does not let a locally connected continuum to admit a mean, [6].

A (metric) continuum X is said to be *arc-like* provided that for each $\varepsilon > 0$ it has an ε-chain cover; or, equivalently, if it is the inverse limit of an inverse sequence of arcs with surjective bonding mappings.

Let an inverse sequence $\{X_n, f_n : n \in \mathbb{N}\}$ be given each coordinate space X_n of which admits a mean $\mu_n : X_n \times X_n \to X_n$ such that for each $n \in \mathbb{N}$ the functional equation $f_n(\mu_{n+1}(x, y)) = \mu_n(f_n(x), f_n(y))$ is satisfied for all $x, y \in X_{n+1}$. Then the inverse limit space $X = \varprojlim \{X_n, f_n : n \in \mathbb{N}\}$ admits a mean $\mu : X \times X \to X$ defined by $\mu(\{x_n\}, \{y_n\}) = \{\mu_n(x_n, y_n)\}$. Some special results concerning this concept are in [9] and [13]. As an answer in the negative to a question whether every mean on an arc-like continuum is an inverse limit mean, [9], a suitable example showing that inverse limit means are not preserved under homeomorphisms has been constructed in [34].

In connection with the main result of [6] that the $\sin(1/x)$-curve does not admit any mean, P. Bacon asked the following.

Questions [6, p. 13]**.** 1) Is the arc the only arc-like continuum that admits a mean? 2) Is the arc the only continuum containing an open dense half-line that admits a mean?

After more than thirty years, the questions still remain unanswered. However, a sequence of important partial answers has been obtained.

The above mentioned result of Bacon (that the $\sin(1/x)$-curve does not admit any mean) has been essentially extended in [10], where some criteria are obtained for the existence as well as for the non-existence of means on continua (the non-existence criterium is also presented in [21, Section 76, p. 374-376]). A further generalization was obtained in [23]. It runs as follows.

Two points a and b of an arc-like continuum are called *opposite end points* of the continuum provided that for each $\varepsilon > 0$ there is an ε-chain cover of the continuum such that only the first link of the chain contains a and only the last link of the chain contains b. Let a continuum X contain an arc-like continuum A with opposite end points a and b of A. A sequence $\{A_n : n \in \mathbb{N}\}$ of subcontinua of X is called a *folding sequence with respect to the point a* provided that for each $n \in \mathbb{N}$ there are two subcontinua P_n and Q_n of A_n such that

$$A_n = P_n \cup Q_n, \quad \mathrm{Lim}\,(P_n \cap Q_n) = \{a\}, \quad \text{and} \quad \mathrm{Lim}\, P_n = \mathrm{Lim}\, Q_n = A.$$

Theorem [23, p. 99]**.** *Let a hereditarily unicoherent continuum X contain an arc-like subcontinuum A with opposite end points a and b of A. If there exist folding sequences $\{A_n\}$ and $\{B_n\}$ with respect to a and b correspondingly, then X admits no mean.*

The concept of a folding sequence is a generalization of the concept of type N [32, p. 837] which in turn generalizes the concept of a zigzag [18, p. 78] and is related to the notion of a bend set [26, p. 548] (These concept were exploited to obtain some criteria for noncontractibility and nonselectibility of *dendroids* (i.e., hereditarily unicoherent and arcwise connected continua) as well as for non-existence of means on these curves. For details see [16, p. 23-32] and [17, p. 496-498].

The above theorem does not apply to hereditarily indecomposable continua, because it assume the existence of decomposable subcontinua. The non-existence of means on the pseudo-arc (and on each hereditarily indecomposable circle-like continuum) follows from the following result, that is shown also in [23].

Theorem [23, p. 102]**.** *If a hereditarily indecomposable contains a pseudo-arc, then it admits no mean.*

Another famous arc-like continuum is the simplest indecomposable continuum D [25, Fig. 4, p. 205] also called the buckethandle continuum or the Brouwer-Janiszewski-Knaster continuum. It can be defined as the inverse limit of arcs with tent bonding mappings. D has exactly one end point, each of its proper subcontinua in an arc, and it again is an example to which the above theorem (on folding sequences of arcs) does not apply. Answering my question [12] A. Illanes has shown that D does not admit any mean [20]. Similarly constructed indecomposable continua with k end points (where $k \geq 2$; for $k = 3$ see [19, p. 142] and [31, 1.10, p. 7]) also do not admit any mean, [15, Corollary 3.15]. Recently D. P. Bellamy [11] presented an outline of a proof that each Knaster-type continuum (i.e., the inverse limit of arcs with open bonding mappings) different from an arc admits no mean.

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