THE PLANE FIXED-POINT PROBLEM

Charles L. Hagopian

Does every nonseparating plane continuum have the fixed-point property? This is the plane fixed-point problem. It has been called the most interesting outstanding problem in plane topology [Bi2]. A positive answer would provide a natural generalization to the 2-dimensional version of the Brouwer fixed-point theorem.

A space S has *the fixed-point property* if for every map (continuous function) f of S into S there exists a point x of S such that $f(x) = x$. A *continuum* is a nondegenerate compact connected metric space. A continuum in the plane that has only one complementary domain is a *nonseparating plane continuum*. Every nonseparating plane continuum is the intersection of a nested sequence of topological disks.

To summarize related results, suppose $\mathcal{C}$ is a nonseparating plane continuum and f is a fixed-point-free map of $\mathcal{C}$ into $\mathcal{C}$. Ayers [Ay] in 1930 proved $\mathcal{C}$ is not locally connected if f is a homeomorphism. In 1932 Borsuk [Bo] proved $\mathcal{C}$ cannot be locally connected (even if f is not a homeomorphism). He accomplished this by showing that every locally connected nonseparating plane continuum is a retract of a disk. Stallings and Borsuk [St] pointed out that the plane fixed-point problem would be solved if it could be shown that every nonseparating plane continuum is an almost continuous retract of a disk. This approach was eliminated by Akis in [A1].

Hamilton [Ha1] in 1938 proved the boundary of $\mathcal{C}$ is not hereditarily decomposable if f is a homeomorphism. Bell [B1], Sieklucki [S], and Illidais [I] in 1967-1970 independently proved the boundary of $\mathcal{C}$ contains an indecomposable continuum that is left invariant by f. The methods used to establish this theorem led to (but did not answer) the following questions. Can the plane fixed-point problem be solved by digging a simple dense canal in a disk? Can f^2 be fixed-point free?

In 1971 Hagopian [H1] proved $\mathcal{C}$ is not arcwise connected. Hagopian [H5] in 1996 improved this theorem by showing that an arcwise connected plane continuum has the fixed-point property if and only if its fundamental group is trivial.

It is not known if the fixed-point-free map f can be a homeomorphism. Bell [B2] in 1978 proved f cannot be a homeomorphism that is extendable to the plane. Akis [A2] and Bell [B3] proved f is not a map that has an analytic extension to the plane. In 1988 Hagopian [H3] proved f cannot send each arc-component of $\mathcal{C}$ into itself. Hence f is not a deformation. Must f permute every arc-component of $\mathcal{C}$?

In 1951 Hamilton [Ha2] proved $\mathcal{C}$ is not chainable. We do not know if $\mathcal{C}$ can be triod-like [M1-2]. More generally, can $\mathcal{C}$ be tree-like [Bi1, p.653]? Bellamy [Be] in 1979 defined a nonplanar tree-like continuum that admits a fixed-point-free map (also see [OR1-2] and [Mi2-5]). Using this example and an inverse limit technique of Fugate and Mohler [FM], Bellamy [Be, p.12] defined a second tree-like continuum M that admits a fixed-point-free homeomorphism. It is not known if M can be embedded in the plane. Note that such an embedding would solve the plane fixed-point problem. Every proper subcontinuum of Bellamy's continuum M is an arc. This motivates another open question. Must a nonseparating plane continuum with only arcs for proper subcontinua have the fixed-point property?

In 1990 Minc [Mi1] proved $\mathcal{C}$ is not weakly chainable (a continuous image of a chainable continuum). Minc [Mi4] in 1999 defined a weakly chainable tree-like continuum that does not have the fixed-point property.

Kuratowski [Ku1] defined a continuum K to be of *type* λ if K is irreducible and every indecomposable continuum in K is a continuum of condensation. Every continuum K of type λ admits a unique montone upper semi-continuous decomposition to an arc with the property that each element of the decomposition has void interior relative to K [Ku2, Th.3, p.216]. The elements of this decomposition are called *tranches*.

Can $\mathcal{C}$ be a continuum of type λ with the property that each of its tranches has the fixed-point property? In answer to a question of Gordh [L, Prob.43, p.371], Hagopian [H6] recently defined a nonplanar continuum $\mathcal{M}$ of type λ such that each tranch of $\mathcal{M}$ has the fixed-point property and $\mathcal{M}$ does not.

A fundamental exposition on the plane fixed-point problem is given in [KW, pp.66 and 145] (also see [Bi3], [H2], and [H4]).

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Department of Mathematics
California State University, Sacramento
Sacramento, California 95819-6051

hagopian@csus.edu