The arc is the only chainable continuum admitting a mean

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Abstract

Let X be a metric continuum. A *mean on* X is a continuous function $\mu : X \times X \to X$ such that for each $x, y \in X$, $\mu(x, y) = \mu(y, x)$ and $\mu(x, x) = x$. In this paper we prove that if X is chainable and admits a mean, then X is an arc. This answers a question stated by Philip Bacon in 1970.

Introduction

Troughout this paper the letter X will denote a metric continuum. A *mean on* X is a continuous function $\mu : X \times X \to X$ such that for each $x, y \in X$, $\mu(x, y) = \mu(y, x)$ and $\mu(x, x) = x$. A *chain in* X is a sequence $\mathcal{U} = \{U_1, \ldots, U_n\}$ of open subsets of X such that $X = U_1 \cup \ldots \cup U_n$ and $U_i \cap U_j \neq \emptyset$ if and only if $|i - j| \leq 1$. Given a positive number ε, the chain $\mathcal{U} = \{U_1, \ldots, U_n\}$ is said to be an ε-*chain* provided that diameter$(U_i) < \varepsilon$ for each $i \in \{1, \ldots, n\}$. We say that X is *chainable* provided that, for each $\varepsilon > 0$ there exists an ε-chain in X.

The problem of determining which continua X admit means has been studied by a number of authors. In [2] P. Bacon proved that if a continuum X admits a mean then X is unicoherent, he also showed ([1]), answering a question by A. D. Wallace ([15]), that the $\sin(\frac{1}{x})$-continuum admits no means. This was the first example of an acyclic continuum admitting no means. More information about means can be found in [6], [9, Section 76] and [10].

In [1] P. Bacon posed the following questions: (1) Is the arc the only chainable continuum that admits a mean? (2) Is the arc the only continuum containing an open dense half-line that admits a mean?

Question (2) has been recently answered, in the positive, by the first named author ([7]). With respect to question (1), answering a question by J. J. Charatonik, the first named author, showed that the simplest indecomposable continuum (also called the buckethandle continuum or the Brouwer-Janiszewski-Knaster continuum) does not admit means ([8]). Recently, D. P. Bellamy ([4])

has shown that each Knaster-type continuum (i.e., the inverse limit of arcs with open bonding mappings) different from the arc admits no mean. Some partial answers to question (1) can be obtained by using the results contained in the papers [3], [5] and [11]. In this paper we give the final answer to question (1) by showing the following.

Theorem 1. If X is chainable and X admits a mean, then X is an arc.

This paper is devoted to prove Theorem 1.

A property in the plane

We denote by $\mathbb{R}^2$ the Euclidean plane. Given points $p, q \in \mathbb{R}^2$, where $p \neq q$, let pq be the convex segment in $\mathbb{R}^2$ joining p and q.

Theorem 2. Let $p = (0, 0)$, $q = (1, 0)$ and $r = (\frac{1}{2}, 1)$ in $\mathbb{R}^2$. Let Δ be the convex triangle in $\mathbb{R}^2$ with vertices p, q and r. Suppose that H and K are closed disjoint subsets of Δ such that $pr \cup rq \subset H$ and $K \cap pq \neq \emptyset$. Then there exists an arc α in Δ, with end points p and q such that $\alpha \cap K = \emptyset$ and $\alpha \cap H \subset pq$.

Proof. Let $V = \Delta - H$. Then V is an open subset of Δ and $K \subset V$. Let $K_0 = K \cap pq$. Since the components of V are open in Δ and they cover the nonempty compact set K_0, there exist $n \in \mathbb{N}$ and components $V_1, \ldots, V_n$ of V such that $K_0 \subset V_1 \cup \ldots \cup V_n$ and K_0 intersects each V_i.

For each $i \in \{1, \ldots, n\}$, let $K_i = K \cap V_i$. Notice that K_i is compact. Let $m_i = \min\{x \in [0, 1] : (x, 0) \in K_i\}$ and $M_i = \max\{x \in [0, 1] : (x, 0) \in K_i\}$. Since $\emptyset \neq K_0 \cap V_i \subset K \cap pq \cap V_i$, m_i and M_i are well defined. Since V_i is arcwise connected, there exists a continuous function $\gamma_i : [0, 1] \to V_i$ such that $\gamma_i(0) = (m_i, 0)$ and $\gamma_i(1) = (M_i, 0)$.

Since $r \notin V_i$, we can apply Theorem 2 of [12, §57, III, p. 438], to the closed sets $\Delta - V_i$ and $K_i \cup \operatorname{Im} \gamma_i$ and the points $r \in \Delta - V_i$ and $(m_i, 0) \in K_i \cup \operatorname{Im} \gamma_i$, then there exists a locally connected subcontinuum C_i of Δ such that $C_i \subset V_i - (K_i \cup \operatorname{Im} \gamma_i)$ and C_i separates r and $(m_i, 0)$ in Δ. Thus C_i intersects the connected set $rp \cup p(m_i, 0)$. Since $rp \cap V_i = \emptyset$, there exists a point $p_i = (u_i, 0) \in p(m_i, 0) \cap C_i$. Similarly, there exists a point $q_i = (v_i, 0) \in (m_i, 0)q \cap C_i$. Since p, $(m_i, 0)$ and q do not belong to C_i, we have that $0 < u_i < m_i < v_i < 1$. Let $\alpha_i : [0, 1] \to C_i$ be a continuous one-to-one function such that $\alpha_i(0) = (u_i, 0)$ and $\alpha_i(1) = (v_i, 0)$, we may assume that $\operatorname{Im} \alpha_i \cap p(m_i, 0) = \{(u_i, 0)\}$ and $\operatorname{Im} \alpha_i \cap (m_i, 0)q = \{(v_i, 0)\}$. Thus $\operatorname{Im} \alpha_i$ intersects the boundary of Δ only at the points $(u_i, 0)$ and $(v_i, 0)$ and we can apply the lemma of the θ-curve ([12, § 61, II, Theorem 2, p. 511]) and conclude that $\Delta - \operatorname{Im} \alpha_i$ has exactly two components D_i and E_i, where D_i and E_i are the respective component of $\Delta - \operatorname{Im} \alpha_i$ which contain the connected sets $F_i = p_iq_i - \{p_i, q_i\}$ and $G_i = p_ip \cup pr \cup rq \cup qq_i - \{p_i, q_i\}$. Since $F_i \cup \operatorname{Im}\gamma_i$ is a connected subset of $\Delta - \operatorname{Im}\alpha_i$, $F_i \cup \operatorname{Im}\gamma_i \subset D_i$, So $(M_i, 0) \in D_i \cap pq \subset p_iq_i$. This implies that $M_i < v_i$. Notice that $\operatorname{Im}\alpha_i \subset V_i - K_i = V_i - (K \cap V_i) \subset V_i - K$, so $\operatorname{Im}\alpha_i \cap (K \cup H) = \emptyset$.

Let $i, j \in \{1, \ldots, n\}$ such that $i \neq j$. If $u_j \in [u_i, v_i]$, then $F_i \cup \operatorname{Im}\alpha_j$ is a connected subset of $\Delta - \operatorname{Im}\alpha_i$, so $q_j \in (F_i \cup \operatorname{Im}\alpha_j) \cap pq \subset D_i \cap pq \subset p_iq_i$. Thus $v_j \in [u_i, v_i]$. Similarly, it can be shown that if $v_j \in [u_i, v_i]$, then $u_j \in [u_i, v_i]$. We have shown that $[u_i, v_i]$ and $[u_j, v_j]$ are disjoint or one is contained in the other. Therefore, taking the maximal intervals of the form $[u_i, v_i]$, there exist $m \in \mathbb{N}$ and $i_1, \ldots, i_m \in \{1, \ldots, n\}$ such that $0 < u_{i_1} < v_{i_1} < u_{i_2} < v_{i_2} < \ldots < u_{i_m} < v_{i_m} < 1$ and $[u_{i_1}, v_{i_1}] \cup [u_{i_2}, v_{i_2}] \cup \ldots \cup [u_{i_m}, v_{i_m}] = [u_1, v_1] \cup [u_2, v_2] \cup \ldots \cup [u_m, v_m]$.

Given a point $w = (u, 0) \in K \cap pq = K_0$, there exists $i \in \{1, \ldots, n\}$ such that $w \in K_i$. Thus $u \in [m_i, M_i] \subset (u_i, v_i) \subset (u_{i_1}, v_{i_1}) \cup (u_{i_2}, v_{i_2}) \cup \ldots \cup (u_{i_m}, v_{i_m})$. This proves that $K \cap pq \subset ((u_{i_1}, v_{i_1}) \cup (u_{i_2}, v_{i_2}) \cup \ldots \cup (u_{i_m}, v_{i_m})) \times \{0\}$.

Let $\beta : [0, 1] \to \Delta$ be the continuous, one-to-one function such that $\beta(0) = p$, $\beta(1) = q$ and $\beta([0, \frac{1}{2m+1}]) = pp_{i_1}$, $\beta([\frac{1}{2m+1}, \frac{2}{2m+1}]) = \operatorname{Im}\alpha_{i_1}$, $\beta([\frac{2}{2m+1}, \frac{3}{2m+1}]) = q_{i_1}p_{i_2}$, $\beta([\frac{3}{2m+1}, \frac{4}{2m+1}]) = \operatorname{Im}\alpha_{i_2}$, $\beta([\frac{4}{2m+1}, \frac{5}{2m+1}]) = q_{i_2}p_{i_3}$,...,$\beta([\frac{2m-1}{2m+1}, \frac{2m}{2m+1}]) = \operatorname{Im}\alpha_{i_m}$, $\beta([\frac{2m}{2m+1}, 1]) = q_{i_m}q$.

Finally, let $\alpha = \operatorname{Im}\beta$. It is easy to check that α has the required properties. ■

PL mappings

A continuous function $f : [0, 1] \longrightarrow [0,\ 1]$ is called a *PL mapping (piecewise linear mapping)*, provided that there exists a partition $P : 0 = t_0 < t_1 < \ldots < t_n = 1$ of $[0, 1]$ such that, for each $i \in \{1, \ldots, n\}$ and each $t \in [t_{i-1}, t_i]$, $f(t) = \frac{t-t_{i-1}}{t_i-t_{i-1}} f(t_i) + \frac{t_i-t}{t_i-t_{i-1}} f(t_{i-1})$ in this case we say that f *is supported by* P. It is easy to see that the class of PL mappings is closed under compositions. A PL mapping f is said to be *a jump mapping* provided that $f(0) = 0$ and $f(1) = 1$.

The following theorem can be proved with the techniques of the paper [14]. We include its proof here for completeness.

Theorem 3. If f and g are jump mappings, then there exist jump mappings α and β such that $f \circ \alpha = g \circ \beta$.

Proof. Let $A = \{(x, y) \in [0, 1]^2 : f(x) = g(y)\}$. The set A is a compact subset of $[0, 1]^2$ such that $(0, 0), (1, 1) \in A$. Let L be the component of A such that $(0, 0) \in L$.

We are going to prove that $(1,1) \in L$. Suppose to the contrary that $(1,1) \notin L$. Then (see [13, Theorem 5.2, p. 72]) there exist compact disjoint subsets H and K of $[0,1]^2$ such that $A = H \cup K$, $(0,0) \in H$ and $(1,1) \in K$. By [12, §57, III, Theorem 2, p. 438], there exists a separator C between $(0,0)$ and $(1,1)$ which is a locally connected continuum disjoint from A. Thus there exist points $(x', y') \in C \cap ((\{0\} \times [0,1]) \cup ([0,1] \times \{1\}))$ and $(u', v') \in C \cap (([0,1] \times \{0\}) \cup (\{1\} \times [0,1]))$. Notice that $f(x') \leq g(y')$ and $f(u') \geq g(v')$. Since C is connected, there exists a point $(t,s) \in C$ such that $f(t) = g(s)$. This implies that $(t,s) \in C \cap A$, a contradiction. Hence $(1,1) \in L$.

Since f and g are PL mappings, there exists a partition $0 = t_0 < t_1 < \ldots < t_n = 1$ such that, for each $i \in \{1, \ldots, n\}$ and each $t \in [t_{i-1}, t_i]$, $f(t) = \frac{t-t_{i-1}}{t_i-t_{i-1}} f(t_i) + \frac{t_i-t}{t_i-t_{i-1}} f(t_{i-1})$ and $g(t) = \frac{t-t_{i-1}}{t_i-t_{i-1}} g(t_i) + \frac{t_i-t}{t_i-t_{i-1}} g(t_{i-1})$.

It is easy to prove that, if $i, j \in \{1, \ldots, n\}$, $x, u \in [t_{i-1}, t_i]$, $y, v \in [t_{j-1}, t_j]$, $f(x) = g(y)$ and $f(u) = g(v)$, then the segment $(x,y)(u,v)$ is contained in A.

Let $p = (x,y) \in A$, now we prove that there exists $\varepsilon_p > 0$ such that if (u,v) belongs to the set $D(\varepsilon_p, p) = A \cap ((x - \varepsilon_p, x + \varepsilon_p) \times (y - \varepsilon_p, y + \varepsilon_p))$, then the segment $(x,y)(u,v)$ is contained in A. In order to prove this claim, by the paragraph above, it is enough to take $\varepsilon_p > 0$ such that if $u \in [x - \varepsilon_p, x + \varepsilon_p] \cap [0,1]$ and $v \in [y - \varepsilon_p, y + \varepsilon_p] \cap [0,1]$, then both points x and u belong to a set of the form $[t_{i-1}, t_i]$ and both points v and y belong to a set of the form $[t_{j-1}, t_j]$.

By the connectedness of the segments of the form (x,y) and (u,v) the claim proved in the paragraph above implies that, if $p = (x,y) \in L$, and $\varepsilon_p > 0$ is as before, then for each $(u,v) \in D(\varepsilon_p, p)$, the segment $(x,y)(u,v)$ is contained in L. Since the family $\{D(\varepsilon_p, p) : p \in L\}$ is an open cover of L and L is connected, it follows that L is connected by polygonals. In particular, there exist $m \in \mathbb{N}$ and points $p_0 = (u_0, v_0)$, $p_1 = (u_1, v_1), \ldots, p_m = (u_m, v_m)$ in L such that $p_0 = (0,0)$, $p_m = (1,1)$ and $p_0p_1 \cup p_1p_2 \cup \ldots \cup p_{m-1}p_m \subset L$.

Define $\alpha, \beta : [0,1] \to [0,1]$ by $\alpha(t) = (mt - i + 1)u_i + (i - mt)u_{i-1}$ and $\beta(t) = (mt - i + 1)v_i + (i - mt)v_{i-1}$, if $t \in [\frac{i-1}{m}, \frac{i}{m}]$. Clearly, α and β have the required properties. ■

Theorem 4. If f is a PL mapping such that $f(1) = 1$ and g is a jump mapping, then there exist a jump mapping α and a PL mapping β such that $\beta(1) = 1$ and $f \circ \alpha = g \circ \beta$.

Proof. In the case that $f(0) = 0$, both mappings f and g are jump mappings, so the existence of α and β follows from Theorem 3. Thus, suppose that $f(0) > 0$.

Let $h, k : [0,1] \to [0,1]$ be the mappings given by

$$h(t) = \begin{cases} 2tf(0), & \text{if } t \in [0, \frac{1}{2}], \\ f(2t-1), & \text{if } t \in [\frac{1}{2}, 1], \end{cases}$$

and

$$k(t) = \begin{cases} 0, & \text{if } t \in [0, \frac{1}{2}], \\ g(2t-1), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

Clearly, h and k are jump mappings. By Theorem 3, there exist jump mappings γ and λ such that $h \circ \gamma = k \circ \lambda$. Let $s_0 = \max \gamma^{-1}(\frac{1}{2})$. Since $\gamma(1) = 1$, $0 < s_0 < 1$.

Define $\alpha, \beta : [0,1] \to [0,1]$ by $\alpha(t) = 2\gamma(t + (1-t)s_0) - 1$ and $\beta(t) = \max\{2\lambda(t + (1-t)s_0) - 1, 0\}$. For each $t \in [0,1]$, since $s_0 \leq t + (1-t)s_0 \leq 1$ and $\gamma(1) = 1$, by the definition of s_0, $\gamma(t + (1-t)s_0) \in [\frac{1}{2}, 1]$. Thus α is a well defined jump mapping, β is a PL mapping and $\beta(1) = 1$.

In order to check that $f \circ \alpha = g \circ \beta$, let $t \in [0,1]$. If $\lambda(t + (1-t)s_0) \geq \frac{1}{2}$, then $2\lambda(t + (1-t)s_0) - 1 \geq 0$, so $g(\beta(t)) = g(2\lambda(t + (1-t)s_0) - 1) = k(\lambda(t + (1-t)s_0)) = h(\gamma(t + (1-t)s_0)) = f(2(\gamma(t + (1-t)s_0) - 1) = f(\alpha(t))$. Thus $g(\beta(t)) = f(\alpha(t))$. And in the case that $\lambda(t+(1-t)s_0) \leq \frac{1}{2}$, $g(\beta(t)) = g(0) = 0$. Notice that $h(\gamma(t+(1-t)s_0)) = k(\lambda(t+(1-t)s_0)) = 0$. On the other hand, since $\gamma(t+(1-t)s_0) \in [\frac{1}{2}, 1]$, $0 = h(\gamma(t+(1-t)s_0) = f(2\gamma(t+(1-t)s_0)-1) = f(\alpha(t))$. Hence $g(\beta(t)) = 0 = f(\alpha(t))$. In both cases $g(\beta(t)) = f(\alpha(t))$. Therefore, $f \circ \alpha = g \circ \beta$. ■

Theorem 5. If f is a PL mapping such that $f(0) = 0$ and g is a jump mapping, then there exist a jump mapping α and a PL mapping β such that $\beta(0) = 0$ and $f \circ \alpha = g \circ \beta$.

Proof. Let $f_1, g_1 : [0,1] \to [0,1]$ be given by $f_1(t) = 1 - f(1-t)$ and $g_1(t) = 1 - g(1-t)$. Then f_1 is a PL mapping such that $f_1(1) = 1$ and g_1 is a jump mapping. By Theorem 4, there exist a jump mapping α_1 and a PL mapping β_1 such that $\beta_1(1) = 1$ and $f_1 \circ \alpha_1 = g_1 \circ \beta_1$.

Define $\alpha, \beta : [0,1] \to [0,1]$ by $\alpha(t) = 1 - \alpha_1(1-t)$ and $\beta(t) = 1 - \beta_1(1-t)$. Then α is a jump mapping and β is a PL mapping such that $\beta(0) = 0$. Moreover, for each $t \in [0,1]$, $f(\alpha(t)) = f(1 - \alpha_1(1-t)) = 1 - f_1(\alpha_1(1-t)) = 1 - g_1(\beta_1(1-t)) = g(1 - \beta_1(1-t)) = g(\beta(t))$. Therefore, $f \circ \alpha = g \circ \beta$. ■

Theorem 6. Let $P : 0 = t_0 < t_1 < \ldots < t_n = 1$ and $Q : 0 = s_0 < s_1 < \ldots < s_m = 1$ be partitions of $[0,1]$. Let $f, g : [0,1] \to [0,1]$ be PL mappings such that f is supported by P and g is supported by Q. Suppose that there exists a function $\sigma : \{0, 1, \ldots, m\} \to \{0, 1, \ldots, n\}$ such that $\sigma(0) = 0$, $\sigma(m) = n$, $f(t_{\sigma(i)}) = g(s_i)$ for each $i \in \{0, 1, \ldots, m\}$ and $|\sigma(i) - \sigma(i-1)| \leq 1$ for each $i \in \{1, \ldots, m\}$. Then there exists a jump mapping α such that $f \circ \alpha = g$.

Proof. Let $\alpha : [0,1] \to [0,1]$ be the PL mapping defined by the conditions $\alpha(s_i) = t_{\sigma(i)}$ for each $i \in \{0,1,\ldots,m\}$. Note that $\alpha(0) = t_0 = 0$ and $\alpha(1) = t_{\sigma(m)} = 1$. Thus α is a jump mapping. In order to check that $f \circ \alpha = g$, let $i \in \{1,\ldots,m\}$ and let $s \in [s_{i-1}, s_i]$. Then $\alpha(s) = \frac{s-s_{i-1}}{s_i-s_{i-1}}\alpha(s_i) + \frac{s_i-s}{s_i-s_{i-1}}\alpha(s_{i-1}) = \frac{s-s_{i-1}}{s_i-s_{i-1}}t_{\sigma(i)} + \frac{s_i-s}{s_i-s_{i-1}}t_{\sigma(i-1)}$. Thus $\alpha(s)$ is a convex combination of the numbers $t_{\sigma(i)}$ and $t_{\sigma(i-1)}$, so one of the following two expresions is a convex combination for $\alpha(s)$ (depending on which inequality: $t_{\sigma(i-1)} \leq t_{\sigma(i)}$ or $t_{\sigma(i)} \leq t_{\sigma(i-1)}$ holds) $\alpha(s) = \frac{\alpha(s)-t_{\sigma(i-1)}}{t_{\sigma(i)}-t_{\sigma(i-1)}}t_{\sigma(i)} + \frac{t_{\sigma(i)}-\alpha(s)}{t_{\sigma(i)}-t_{\sigma(i-1)}}t_{\sigma(i-1)}$, $\alpha(s) = \frac{\alpha(s)-t_{\sigma(i)}}{t_{\sigma(i-1)}-t_{\sigma(i)}}t_{\sigma(i-1)} + \frac{t_{\sigma(i-1)}-\alpha(s)}{t_{\sigma(i-1)}-t_{\sigma(i)}}t_{\sigma(i)}$. By hypothesis, $|\sigma(i) - \sigma(i-1)| \leq 1$. If $\sigma(i) > \sigma(i-1)$, since f is supported by P, $f(\alpha(s)) = \frac{\alpha(s)-t_{\sigma(i-1)}}{t_{\sigma(i)}-t_{\sigma(i-1)}}f(t_{\sigma(i)}) + \frac{t_{\sigma(i)}-\alpha(s)}{t_{\sigma(i)}-t_{\sigma(i-1)}}f(t_{\sigma(i)-1}) = \frac{\alpha(s)-t_{\sigma(i-1)}}{t_{\sigma(i)}-t_{\sigma(i-1)}}f(t_{\sigma(i)}) + \frac{t_{\sigma(i)}-\alpha(s)}{t_{\sigma(i)}-t_{\sigma(i-1)}}f(t_{\sigma(i-1)}) = \frac{\alpha(s)-t_{\sigma(i-1)}}{t_{\sigma(i)}-t_{\sigma(i-1)}}g(s_i) + \frac{t_{\sigma(i)}-\alpha(s)}{t_{\sigma(i)}-t_{\sigma(i-1)}}g(s_{i-1})$. The equality $\frac{s-s_{i-1}}{s_i-s_{i-1}}t_{\sigma(i)} + \frac{s_i-s}{s_i-s_{i-1}}t_{\sigma(i-1)} = \frac{\alpha(s)-t_{\sigma(i-1)}}{t_{\sigma(i)}-t_{\sigma(i-1)}}t_{\sigma(i)} + \frac{t_{\sigma(i)}-\alpha(s)}{t_{\sigma(i)}-t_{\sigma(i-1)}}t_{\sigma(i-1)}$ implies that $\frac{s-s_{i-1}}{s_i-s_{i-1}} = \frac{\alpha(s)-t_{\sigma(i-1)}}{t_{\sigma(i)}-t_{\sigma(i-1)}}$ and $\frac{s_i-s}{s_i-s_{i-1}} = \frac{t_{\sigma(i)}-\alpha(s)}{t_{\sigma(i)}-t_{\sigma(i-1)}}$. Thus $f(\alpha(s)) = \frac{s-s_{i-1}}{s_i-s_{i-1}}g(s_i) + \frac{s_i-s}{s_i-s_{i-1}}g(s_{i-1}) = g(s)$. Hence $f(\alpha(s)) = g(s)$. The case $\sigma(i) < \sigma(i-1)$ is similar. Finally, if $\sigma(i) = \sigma(i-1)$, then $\alpha(s) = t_{\sigma(i)} = t_{\sigma(i-1)}$. Hence, $f(\alpha(s)) = f(t_{\sigma(i)}) = f(t_{\sigma(i-1)}) = g(s_i) = g(s_{i-1})$. Since g is supported by Q, $g(s) = g(s_i)$. Therefore $f(\alpha(s)) = g(s)$. The proof of the theorem is complete. ■

Chains

For a chainable continuum X, with metric d, and a positive number ε, we say that a chain $\mathcal{U} = \{U_1, \ldots, U_n\}$ is a *separated chain in* X provided that U_1 is not contained in $\mathrm{cl}_X(U_2)$, U_n is not contained in $\mathrm{cl}_X(U_{n-1})$ and $\mathrm{cl}_X(U_i) \cap \mathrm{cl}_X(U_j) \neq \emptyset$ if and only if $|i-j| \leq 1$. If, in addition, diameter$(U_i) < \varepsilon$ for each $i \in \{1, \ldots, n\}$, then $\mathcal{U}$ is said to be a separated ε-chain. It is easy to see that if $\mathcal{V} = \{V_1, \ldots, V_m\}$ is a δ-chain in X, then the sequence $\{V_1 \cup V_2, V_3 \cup V_4, \ldots\}$ (the last element in this sequence is V_m, if m is od and, it is $V_{m-1} \cup V_m$, if m is even) is a separated 2δ-chain in X. Thus for each $\varepsilon > 0$ there exists a separated ε-chain in X.

Given a chain $\mathcal{U} = \{U_1, \ldots, U_n\}$, there is a natural order in $\mathcal{U}$ (given by the order of the subindices) which will be denoted with the usual symbols $<$, $>$, $\leq$ and $\geq$. So, if $U, V \in \mathcal{U}$, we define $UV = \bigcup\{W \in \mathcal{U} : U \leq W \leq V\}$, if $U \leq V$ and $UV = \bigcup\{W \in \mathcal{U} : V \leq W \leq U\}$, if $V \leq U$. We also say that an element $W \in \mathcal{U}$ *is between* $U, V \in \mathcal{U}$ provided that $U \leq W \leq V$, if $U \leq V$, and $V \leq W \leq U$, if $V \leq U$.

Given a separated chain $\mathcal{U} = \{U_1, \ldots, U_n\}$ in X, with $n \geq 3$, the *tightness* $t(\mathcal{U})$ of $\mathcal{U}$ is defined as

$$t(\mathcal{U}) = \min\{d(\bigcup\{\text{cl}_X(U) : U < U_i\}, \bigcup\{\text{cl}_X(U) : U_i < U\}) : i \in \{2, \ldots, n-1\}\}$$

Given two separated chains $\mathcal{V}$ and $\mathcal{U} = \{U_1, \ldots, U_n\}$ in X, with $n \geq 3$, we say that $\mathcal{V}$ *ultrarefines* $\mathcal{U}$ provided that: (a) $\mathcal{V}$ is a $\frac{t(\mathcal{U})}{3}$-chain, (b) there exist $V, W \in \mathcal{V}$ such that $V \cap (U_2 \cup \ldots \cup U_n) = \emptyset$, $W \cap (U_1 \cup \ldots \cup U_{n-1}) = \emptyset$ and, (c) for each $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $V \subset U$. Clearly, for each separated chain $\mathcal{U}$ in X and for each $\varepsilon > 0$, there exists a separated ε-chain $\mathcal{V}$ in X such that $\mathcal{V}$ ultrarefines $\mathcal{U}$.

Given two separated chains $\mathcal{V}$ and $\mathcal{U}$ in X such that $\mathcal{V}$ ultrarefines $\mathcal{U}$ and given $U, V \in \mathcal{U}$, with $U \neq V$, we say that $\mathcal{V}$ *folds from V to U* provided that there exist $P, Q, R \in \mathcal{V}$ such that $P < Q < R$ or $R < Q < P$, $RP \subset UV$, $P \cup R \subset V$ and $Q \subset U$. We also say that $\mathcal{V}$ *makes a zigzag between U and V with elements* $P, Q, R, S \in \mathcal{V}$ if $P < Q < R < S$, $SP \subset UV$, $P \cup R \subset U$ and $Q \cup S \subset V$.

Given $\varepsilon > 0$, metric spaces Y and Z, and an onto mapping $f : Y \to Z$, f is said to be an *ε-mapping* provided that diameter$(f^{-1}(z)) < \varepsilon$ for each $z \in Z$.

The following lemma is easy to prove.

Lemma 7. Let X be a chainable continuum and let $\mathcal{U}$ and $\mathcal{V}$ be separated chains in X such that $\mathcal{V}$ ultrarefines $\mathcal{U}$. Then:

(a) If $U, V \in \mathcal{U}$, where $U < V$ and $P,\ Q \in \mathcal{V}$ satisfy $P \cap U \neq \emptyset$ and $Q \cap V \neq \emptyset$, then for each $W \in \mathcal{U}$ such that $U < W < V$, there exists $R \in \mathcal{V}$ such that R is between P and Q, $R \cap W \neq \emptyset$ and the only element of $\mathcal{U}$ which intersects R is W.

(b) If $U, V \in \mathcal{U}$, where $U \neq V$, then there exist $P,\ Q \in \mathcal{V}$ such that $P \subset U$, $Q \subset V$, $PQ \subset UV$ and PQ intersects W for each $W \in \mathcal{U}$ which is between U and V.

(c) If A is a subcontinuum of X, U (resp., V) is the first (resp., last) element of $\mathcal{U}$ intersecting A, then $A \subset UV$ and A intersects each element $W \in \mathcal{U}$ which is between U and V.

(d) If $U, V \in \mathcal{U}$ and $U < V$, then there exists a subcontinuum A of X such that $A \subset \text{cl}_X(UV)$, $A \cap \text{cl}_X(U) \neq \emptyset$ and $A \cap \text{cl}_X(V) \neq \emptyset$.

(e) If $\mathcal{U}$ is an ε-chain, $U, V \in \mathcal{U}$ and $\text{cl}_X(U) \cap \text{cl}_X(V) = \emptyset$, then there exists an onto ε-mapping $\varphi : \text{cl}_X(UV) \longrightarrow [0, 1]$ such that $\text{cl}_X(U) = \varphi^{-1}(0)$ and $\text{cl}_X(V) = \varphi^{-1}(1)$.

Two basic results

Throughout this paper the letter X will denote a continuum, with metric d, we define the metric D on $X \times X$ by $D((u, v), (x, y)) = \frac{1}{2}(d(u, x) + d(v, y))$. Given two nonempty subsets K and L of X we define $d(K, L) = \inf\{d(x, y) : x \in K$ and $y \in L\}$. For subsets $K, L \subset X \times X$, the symbol $D(K, L)$ is defined in a similar way.

Given a chainable continuum X, a mean $\mu : X \times X \to X$, a separated chain $\mathcal{U}$ and elements U and V of $\mathcal{U}$, we define $\mathfrak{D}(\mathrm{cl}_X(U)) = \{(u, u) \in X \times X : u \in \mathrm{cl}_X(U)\}$ and

$$\mathcal{D}(U, V) = \bigcup\{E : E \text{ is a component of } (\mathrm{cl}_X(UV) \times \mathrm{cl}_X(UV)) \cap \mu^{-1}(\mathrm{cl}_X(U)) \text{ and } E \cap \mathfrak{D}(\mathrm{cl}_X(U)) \neq \emptyset\}.$$

Theorem 8. Let X be a chainable continuum, $\mu : X \times X \to X$ a mean, $\mathcal{U}$ a separated chain in X and $U, V \in \mathcal{U}$. Suppose that $\mathrm{cl}_X(U) \cap \mathrm{cl}_X(V) = \emptyset$ and $\mathcal{D}(U, V) \cap ((\mathrm{cl}_X(UV) \times \mathrm{cl}_X(V)) \cup (\mathrm{cl}_X(V) \times \mathrm{cl}_X(UV)) = \emptyset$. Then there exists $\eta > 0$ such that, if $\mathcal{V}$ is a separated η-chain in X, then $\mathcal{V}$ ultrarefines $\mathcal{U}$ and $\mathcal{V}$ does not fold from V to U.

Proof. Let $L = \mathrm{cl}_X(UV) \times \mathrm{cl}_X(UV)$, $J = \mathfrak{D}(\mathrm{cl}_X(U))$ and $M = (\mathrm{cl}_X(UV) \times \mathrm{cl}_X(V)) \cup (\mathrm{cl}_X(V) \times \mathrm{cl}_X(UV))$. Let $N = (L \cap \mu^{-1}(\mathrm{cl}_X(U))) \cup M$. Then N is a compact subset of L. Given a component E of N, we claim that either $E \cap J = \emptyset$ or $E \cap M = \emptyset$. Suppose to the contrary that $E \cap J \neq \emptyset$ and $E \cap M \neq \emptyset$. Fix a point $p \in E \cap J$. Since $\mathrm{cl}_X(U) \cap \mathrm{cl}_X(V) = \emptyset$, $M \cap J = \emptyset$, so $p \notin M$. Let C be the component of $E - M$ containing p. Since $E - M$ is a proper nonempty subset of the continuum E, by [13, Theorem 5.6, p. 74], $\emptyset \neq \mathrm{cl}_E(C) \cap \mathrm{bd}_E(E - M) \subset \mathrm{cl}_E(C) \cap M$. On the other hand, since $C \subset L \cap \mu^{-1}(\mathrm{cl}_X(U))$, there exists a component F of $L \cap \mu^{-1}(\mathrm{cl}_X(U))$ such that $C \subset F$. Then $\emptyset \neq \mathrm{cl}_E(C) \cap M \subset F \cap M$. Since $p \in F \cap J$, $F \subset \mathcal{D}(U, V)$. Thus $\mathcal{D}(U, V) \cap M \neq \emptyset$, contrary to our assumption on $\mathcal{D}(U, V)$. We have proved that $E \cap J = \emptyset$ or $E \cap M = \emptyset$. Therefore, no component of N intersects both sets $N \cap J$ and M. By [13, Theorem 5.2, p. 72], there exist disjoint compact sets K and G such that $N = K \cup G$, $N \cap J \subset K$ and $M \subset G$. For each point $u \in U$, $(u, u) \in N \cap J$. Thus $N \cap J \neq \emptyset$.

Fix $0 < \eta < D(K, L)$ such that, if $\mathcal{V}$ is a separated η-chain in X, then $\mathcal{V}$ has at least three elements and $\mathcal{V}$ ultrarefines $\mathcal{U}$.

Let $\mathcal{V}$ be a separated η-chain in X. We are going to prove that $\mathcal{V}$ does not fold from V to U.

Suppose to the contrary that that $\mathcal{V}$ folds from V to U. Then there exist $P, Q, R \in \mathcal{V}$ such that $P < Q < R$ or $R < Q < P$, $PR \subset UV$, $P \cup R \subset V$ and $Q \subset U$. Let $\delta > 0$ be such that $\delta < \eta$ and, if $D((u, v), (x, y)) < \delta$, then $d(\mu(u, v), \mu(x, y)) < t(\mathcal{V})$.

Let $\mathcal{W}$ be a separated δ-chain such that $\mathcal{W}$ ultrarefines $\mathcal{V}$. By Lemma 7 (b), there exist $S, T \in \mathcal{W}$ such that $S \subset P$, $T \subset R$, $ST \subset PR$ and ST intersects W

for each $W \in \mathcal{V}$ which is between P and R. By Lemma 7 (d), there exists an onto δ-mapping $\varphi : \mathrm{cl}_X(ST) \longrightarrow [0,\ 1]$ such that $\mathrm{cl}_X(S) = \varphi^{-1}(0)$ and $\mathrm{cl}_X(T) = \varphi^{-1}(1)$. Let $\psi : \mathrm{cl}_X(ST)\times \mathrm{cl}_X(ST) \to [0,1]^2$ be given by $\psi(x,y) = (\varphi(x), \varphi(y))$. Then ψ is an onto δ-mapping.

Suppose that there exist points $(u,v), (x,y) \in \mathrm{cl}_X(ST)\times \mathrm{cl}_X(ST)$ such that $(u,v) \in K$, $(x,y) \in G$ and $\psi(u,v) = \psi(x,y)$. Then $D((u,x),(v,y)) < \delta < D(K,\ G)$, a contradiction. We have shown that $\psi((\mathrm{cl}_X(ST)\times \mathrm{cl}_X(ST)) \cap K) \cap \psi((\mathrm{cl}_X(ST)\times \mathrm{cl}_X(ST)) \cap G) = \emptyset$.

We show that the boundary B of $[0,1]^2$ in $\mathbb{R}^2$ is contained in $\psi((\mathrm{cl}_X(ST)\times \mathrm{cl}_X(ST)) \cap G)$. Given a point $(0,s) \in B$, since φ is onto, there exist points $x \in S \subset P \subset V$ and $y \in \mathrm{cl}_X(ST)$ such that $\psi(x,y) = (0,s)$. Thus $(x,y) \in M \subset G$. Hence $\{0\} \times [0,1] \subset \psi((\mathrm{cl}_X(ST)\times \mathrm{cl}_X(ST)) \cap G)$. The rest of points of B can be treated in a similar way. Thus $B \subset \psi((\mathrm{cl}_X(ST)\times \mathrm{cl}_X(ST)) \cap G)$.

Let Δ denote the triangle contained in $[0,1]^2$ with vertices $(0,0), (0,1)$ and $(1,1)$ and let Λ denote the diagonal of $[0,1]^2$. By the choice of S and T, there exists a point $x_0 \in ST \cap Q \subset U$. Then $(x_0,x_0) \in N \cap J \subset K$. Thus $\psi(x_0,x_0) \in \Lambda \cap \psi((\mathrm{cl}_X(ST)\times \mathrm{cl}_X(ST)) \cap K)$.

Hence we can apply Theorem 2 to the triangle Δ, the closed disjoint subsets $H_0 = \psi((\mathrm{cl}_X(ST)\times \mathrm{cl}_X(ST)) \cap G) \cap \Delta$ and $K_0 = \psi((\mathrm{cl}_X(ST)\times \mathrm{cl}_X(ST)) \cap K) \cap \Delta$, since $(\{0\} \times [0,1]) \cup ([0,1] \times \{1\}) \subset H_0$ and $\psi(x_0,x_0) \in K_0 \cap \Lambda$. Thus there exists a one-to-one continuous function $\beta : [0,1] \to \Delta$ such that $\beta(0) = (0,0)$, $\beta(1) = (1,1)$, $\mathrm{Im}\,\beta \cap K_0 = \emptyset$ and $\mathrm{Im}\,\beta \cap H_0 \subset \Lambda$.

Since ψ is a δ-mapping, there exists $\varepsilon > 0$ such that if $A \subset [0,1]^2$ and $\mathrm{diameter}(A) < \varepsilon$, then $\mathrm{diameter}(\psi^{-1}(A)) < \delta$. Since β is uniformly continuous, there exists $\lambda > 0$ such that, if $|t-s| < \lambda$, then $\|\beta(t) - \beta(s)\| < \varepsilon$.

Let $0 = t_0 < t_1 < t_2 < \ldots < t_m = 1$ be a partition of the interval $[0,1]$ such that $t_i - t_{i-1} < \lambda$ for each $i \in \{1,2,\ldots,m\}$. For each $i \in \{0,1,2,\ldots,m\}$, choose an element $(x_i,y_i) \in \mathrm{cl}_X(ST)\times \mathrm{cl}_X(ST)$ such that $\psi(x_i,y_i) = \beta(t_i)$, in the case that $\beta(t_i) \in \Lambda$, $\beta(t_i) = (t,t)$ for some $t \in [0,1]$, since φ is onto, we can choose $x_i \in \mathrm{cl}_X(ST)$ such that $\varphi(x_i) = t$, then $\psi(x_i,x_i) = \beta(t_i)$, so in the case that $\beta(t_i) \in \Lambda$, we can assume that $x_i = y_i$. In particular, since $\beta(0) = (0,0)$ and $\beta(1) = (1,1)$, $x_0 \in \mathrm{cl}_X(S)$ and $x_m \in \mathrm{cl}_X(T)$.

For each $i \in \{0,1,2,\ldots,m\}$, let $p_i = \mu(x_i,y_i)$. Then $p_0 = x_0 = \mu(x_0,x_0) \in \mathrm{cl}_X(S) \subset \mathrm{cl}_X(P)$ and $p_m = x_m = \mu(x_m,x_m) \in \mathrm{cl}_X(T) \subset \mathrm{cl}_X(R)$. Given $i \in \{1,2,\ldots,m\}$, since $t_i - t_{i-1} < \lambda$, $\|\beta(t_i) - \beta(t_{i-1})\| < \varepsilon$. Since (x_i,y_i), $(x_{i-1},y_{i-1}) \in \psi^{-1}(\{\beta(t_i), \beta(t_{i-1})\})$, by the choice of ε, $D((x_i,y_i),(x_{i-1},y_{i-1})) < \delta$. By the choice of δ, $d(p_i,p_{i-1}) < t(\mathcal{V})$.

Since $p_0 \in \mathrm{cl}_X(P)$, $p_m \in \mathrm{cl}_X(R)$, $P < Q < R$ and $d(p_i,p_{i-1}) < t(\mathcal{V})$ for each $i \in \{1,2,\ldots,m\}$, there exists $j \in \{1,2,\ldots,m\}$ such that $p_j \in Q$. Thus $p_j \in$

$\mathrm{cl}_X(U)$ and $(x_j, y_j) \in \mu^{-1}(\mathrm{cl}_X(U)) \cap (\mathrm{cl}_X(ST) \times \mathrm{cl}_X(ST)) \subset \mu^{-1}(\mathrm{cl}_X(U)) \cap L \subset N = K \cup G$. Hence $(x_j, y_j) \in K \cup G$. We know that $\psi(x_j, y_j) \in \mathrm{Im}\,\beta \subset \Delta$ and $\mathrm{Im}\,\beta \cap K_0 = \emptyset$, this implies that $(x_j, y_j) \notin K$. Thus $(x_j, y_j) \in G$. This implies that $\psi(x_j, y_j) \in H_0 \cap \mathrm{Im}\,\beta \subset \Lambda$. By the choice of (x_j, y_j), $x_j = y_j$. Then $x_j = \mu(x_j, y_j) = p_j \in \mathrm{cl}_X(U)$. Thus $(x_j, x_j) \in N \cap J \subset K$. Hence $(x_j, y_j) \in G \cap K$, a contradiction.

We have shown that $\mathcal{V}$ does not fold from V to U. This completes the proof of the theorem. ■

Theorem 9. Let X be a chainable continuum and $\mu : X \times X \to X$ a mean. Then for each $\varepsilon > 0$, there exists $\lambda > 0$ such that, if $\mathcal{U}$ is a separated λ-chain in X and $U, V \in \mathcal{U}$ are such that $d(\mathrm{cl}_X(U), \mathrm{cl}_X(V)) \geq \varepsilon$, then there exists $\eta > 0$ such that for each separated η-chain $\mathcal{V}$, $\mathcal{V}$ ultrarefines $\mathcal{U}$, and $\mathcal{V}$ does not fold from V to U or $\mathcal{V}$ does not fold from U to V.

Proof. Let $\varepsilon > 0$. Since μ is uniformly continuous, there exists $\lambda > 0$ such that, if $D((x, y), (u, v)) < \lambda$, then $d(\mu(x,\ y), \mu(u,\ v)) < \varepsilon$. Let $\mathcal{U}$ be a separated λ-chain and let $U, V \in \mathcal{U}$ be such $d(\mathrm{cl}_X(U), \mathrm{cl}_X(V)) \geq \varepsilon$.

Claim. $\mathcal{D}(U, V) \cap ((\mathrm{cl}_X(UV) \times \mathrm{cl}_X(V)) \cup (\mathrm{cl}_X(V) \times \mathrm{cl}_X(UV)) = \emptyset$ or $\mathcal{D}(V, U) \cap ((\mathrm{cl}_X(UV) \times \mathrm{cl}_X(U)) \cup (\mathrm{cl}_X(U) \times \mathrm{cl}_X(UV)) = \emptyset$.

In order to prove this claim, suppose that it is not true. By Lemma 7 (e), there exists an onto λ-mapping $\varphi : \mathrm{cl}_X(UV) \longrightarrow [0, 1]$ such that $\mathrm{cl}_X(U) = \varphi^{-1}(0)$ and $\mathrm{cl}_X(V) = \varphi^{-1}(1)$. Define $\psi : \mathrm{cl}_X(UV) \times \mathrm{cl}_X(UV) \to [0, 1]^2$ by $\psi(x, y) = (\varphi(x), \varphi(y))$. Then ψ is an onto λ-mapping.

Given a component E of $(\mathrm{cl}_X(UV) \times \mathrm{cl}_X(UV)) \cap \mu^{-1}(\mathrm{cl}_X(U))$ such that $E \cap \mathfrak{D}(\mathrm{cl}_X(U)) \neq \emptyset$, there exists $x_0 \in \mathrm{cl}_X(U)$ such that $(x_0, x_0) \in E$, so $(0, 0) \in \psi(E)$. Since μ is symmetric, E is a symmetric subset of $\mathrm{cl}_X(UV) \times \mathrm{cl}_X(UV)$. This implies that $\psi(\mathcal{D}(U, V))$ is a symmetric subcontinuum of $[0, 1]^2$ containing $(0, 0)$, and by our assumption on $\mathcal{D}(U, V)$, $\psi(\mathcal{D}(U, V))$ intersects the set $[0, 1] \times \{1\}$. Similarly, $\psi(\mathcal{D}(V, U))$ is a symmetric subcontinuum of $[0, 1]^2$ containing $(1, 1)$ and intersecting the set $\{0\} \times [0, 1]$. This implies that $\psi(\mathcal{D}(U, V)) \cap \psi(\mathcal{D}(V, U)) \neq \emptyset$. Take points $(x, y) \in \mathcal{D}(U, V) \subset \mu^{-1}(\mathrm{cl}_X(U))$ and $(u, v) \in \mathcal{D}(V, U) \subset \mu^{-1}(\mathrm{cl}_X(V))$ such that $\psi(x, y) = \psi(u, v)$. Then $D((x, y), (u, v)) < \lambda$ and $d(\mu(x, y), \mu(u, v)) < \varepsilon$. This contradicts the choice of U and V and completes the proof of the claim.

Suppose, without loss of generality, that $\mathcal{D}(U, V) \cap ((\mathrm{cl}_X(UV) \times \mathrm{cl}_X(V)) \cup (\mathrm{cl}_X(V) \times \mathrm{cl}_X(UV)) = \emptyset$.

Let $\eta > 0$ be as in Theorem 8. Hence, if $\mathcal{V}$ is a separated η-chain in X, then $\mathcal{V}$ ultrarefines $\mathcal{U}$ and $\mathcal{V}$ does not fold from V to U. ■

The hereditarily decomposable case

A nondegenerate continuum X is *decomposable* provided that there exist two proper subcontinua A and B of X such that $X = A \cup B$. The continuum X is said to be *hereditarily decomposable* if each nondegenerate subcontinuum of X is decomposable. Given two points $p, q \in X$, we say that X is *irreducible between p and q*, provided that there is no proper subcontinuum of X containing both points p and q.

Given a subcontinuum A of a chainable continuum X and a chain $\mathcal{U}$ in X, we say that two elements U and V of $\mathcal{U}$ *bound* A provided that $A \subset UV$, $U \leq V$ and $\{W \in \mathcal{U} : U \leq W \leq V\}$ is a minimal subchain of $\mathcal{U}$ containing A. Note that $W \cap A \neq \emptyset$ for each $W \in \mathcal{U}$ such that $U \leq W \leq V$. Note also that, if A is not contained in the intersection of two elements of $\mathcal{U}$ then U and V are unique.

Theorem 10. If X is a hereditarily decomposable chainable continuum and X admits a mean, then X is an arc.

Proof. Let $\mu : X \times X \to X$ be a mean. Suppose to the contrary that X is not an arc. By [13, Theorem 12.5, p. 233] there exist two points p and q of X such that X is irreducible between p and q. By [12, Theorem 3, p. 216], there exists a monotone mapping $\varphi : X \longrightarrow [0, 1]$ such that $\varphi(p) = 0$, $\varphi(q) = 1$ and $\mathrm{int}_X(\varphi^{-1}(t)) = \emptyset$ for each $t \in [0, 1]$.

Since X is not an arc, φ is not one-to-one. Thus, there exists $t_0 \in I$ such that $W = \varphi^{-1}(t_0)$ is nondegenerate. Note that $W \subset \mathrm{cl}_X(\varphi^{-1}([0, t_0))) \cup \mathrm{cl}_X(\varphi^{-1}((t_0, 1]))$. So, $W = (\mathrm{cl}_X(\varphi^{-1}([0, t_0)) \cap W) \cup (\mathrm{cl}_X(\varphi^{-1}((t_0, 1])) \cap W)$. Without loss of generality we can assume that $Y = \mathrm{cl}_X(\varphi^{-1}((t_0, 1])) \cap W$ is nondegenerate.

Since X is monotone, each set of the form $\varphi^{-1}([t, 1])$ is a subcontinuum of X, this implies that $\varphi^{-1}((t_0, 1])$ is connected. Since X is chainable, X is hereditarily unicoherent ([13, Theorem 12.2, p. 230]). Thus Y is a subcontinuum of X. Since Y itself is chainable, there exist points $p_0, q_0 \in Y$ such that Y is irreducible between p_0 and q_0. Hence there exists a monotone mapping $\pi : Y \longrightarrow [0, 1]$ such that $\pi(p_0) = 0$, $\pi(q_0) = 1$ and $\mathrm{int}_Y(\pi^{-1}(t)) = \emptyset$ for each $t \in [0, 1]$.

Let $\varepsilon = \frac{1}{4} d(\pi^{-1}([0, \frac{1}{3}]), \pi^{-1}([\frac{2}{3}, 1]))$. By Theorem 9, there exists $\lambda > 0$ such that, if $\mathcal{U}$ is a separated λ-chain and $U, V \in \mathcal{U}$ are such that $d(\mathrm{cl}_X(U), \mathrm{cl}_X(V)) \geq \varepsilon$, then there exists $\eta > 0$ such that for each separated η-chain $\mathcal{V}$, $\mathcal{V}$ ultrarefines $\mathcal{U}$, and $\mathcal{V}$ does not fold from V to U or $\mathcal{V}$ does not fold from U to V.

Let $\delta > 0$ be such that

$$4\delta < \min\{\lambda, d(\pi^{-1}([0, \ \tfrac{2}{3}]), \ \pi^{-1}(1)), d(\pi^{-1}(0), \ \pi^{-1}([\tfrac{1}{3}, \ 1])), \varepsilon\}.$$

Let $\mathcal{U}$ be a separated δ-chain in X. Let $U_{p_0}, U_{q_0} \in \mathcal{U}$ be such that $p_0 \in U_{p_0}$ and $q_0 \in U_{q_0}$. At this point we have two possible orders for $\mathcal{U}$. So, we choose the order that satisfies $U_{p_0} < U_{q_0}$. Given elements $W_1, W_2 \in \mathcal{U}$ such that $p_0 \in W_1$ and $q_0 \in W_2$, we have that diameter$(U_{p_0} \cup W_1)$, diameter$(U_{q_0} \cup W_2) < \frac{\varepsilon}{2}$. If $(U_{p_0} \cup W_1) \cap (U_{q_0} \cup W_2) \neq \emptyset$, then $d(p_0, q_0) < \varepsilon$, contradicting the choice of ε. Therefore, $(U_{p_0} \cup W_1) \cap (U_{q_0} \cup W_2) = \emptyset$. Since $U_{p_0} < U_{q_0}$, we conclude that $W_1 < W_2$. Let $U, V \in \mathcal{U}$ be such that U and V bound Y.

Claim. $d(\mathrm{cl}_X(U), \mathrm{cl}_X(V)) \geq \varepsilon$.

In order to prove the claim, let $U_1, U_2, U_3, U_4, U_5, U_6, U_7, U_8 \in \mathcal{U}$ be such U_1 and U_2 bound $\pi^{-1}(0)$; U_3 and U_4 bound $\pi^{-1}([0, \frac{2}{3}])$; U_5 and U_6 bound $\pi^{-1}([\frac{1}{3}, 1])$ and U_7 and U_8 bound $\pi^{-1}(1)$. Note that we may assume that $U \leq U_3 \leq U_1 \leq U_2 \leq U_4 \leq V$ and $U \leq U_5 \leq U_7 \leq U_8 \leq U_6 \leq V$.

We show that $U_1U_2 \cap U_5U_6 = \emptyset$ and $U_3U_4 \cap U_7U_8 = \emptyset$. Suppose that there exists a point $x \in U_1U_2 \cap U_5U_6$. Then there exist $P, Q \in \mathcal{U}$ such that $x \in P \cap Q$ and $U_1 \leq P \leq U_2$, $U_5 \leq Q \leq U_6$. We can take points $y \in P \cap \pi^{-1}(0)$ and $z \in Q \cap \pi^{-1}([\frac{1}{3}, 1])$. Then $d(y, z) \leq d(y, x) + d(x, z) \leq$ diameter$(P)+$ diameter$(Q) < 2\delta < d(\pi^{-1}(0), \pi^{-1}([\frac{1}{3}, 1])) \leq d(y, z)$, a contradiction. We have shown that $U_1U_2 \cap U_5U_6 = \emptyset$. Similarly, $U_3U_4 \cap U_7U_8 = \emptyset$.

Since $p_0 \in U_1U_2$ and $q_0 \in U_5U_6$, there exist $W_1, W_2 \in \mathcal{U}$ such that $p_0 \in W_1$, $q_0 \in W_2$, $U_1 \leq W_1 \leq U_2$ and $U_5 \leq W_2 \leq U_6$. Then $W_1 < W_2$. Since $U_1U_2 \cap U_5U_6 = \emptyset$, we conclude that $U_2 < U_5$. Similarly, $U_4 < U_7$.

If $U \cap \pi^{-1}([\frac{1}{3}, 1]) \neq \emptyset$, then $U \cap U_5U_6 \neq \emptyset$. This is impossible since $U \leq U_2 < U_5 \leq U_6$. Hence $U \cap \pi^{-1}([\frac{1}{3}, 1]) = \emptyset$. On the other hand, $U \cap Y \neq \emptyset$, so $U \cap \pi^{-1}([0, \frac{1}{3}]) \neq \emptyset$. Similarly, $V \cap \pi^{-1}([\frac{2}{3}, 1]) \neq \emptyset$. If $d(\mathrm{cl}_X(U), \mathrm{cl}_X(V)) < \varepsilon$, then there exist points $x \in \mathrm{cl}_X(U)$ and $y \in \mathrm{cl}_X(V)$ such that $d(x, y) < \varepsilon$. Since diameter$(\mathrm{cl}_X(U))$ and diameter$(\mathrm{cl}_X(V)) < \varepsilon$, we conclude that $d(\pi^{-1}([0, \frac{1}{3}]), \pi^{-1}([\frac{2}{3}, 1])) < 3\varepsilon$, contradicting the definition of ε. Therefore, $d(\mathrm{cl}_X(U), \mathrm{cl}_X(V)) \geq \varepsilon$ and the claim is proved.

Since $\mathcal{U}$ is a separated λ-chain, there exists $\eta > 0$ such that for each separated η-chain $\mathcal{V}$, $\mathcal{V}$ ultrarefines $\mathcal{U}$, and $\mathcal{V}$ does not fold from V to U or $\mathcal{V}$ does not fold from U to V.

Since $\bigcap\{\mathrm{cl}_X(\varphi^{-1}((t_0, t_0 + \frac{1}{n}])) : n \in \mathbb{N}\}$ is contained in Y, there exists $n \in \mathbb{N}$ such that $\varphi^{-1}((t_0, t_0 + \frac{1}{n}]) \subset UV$. Fix points $u_1 \in Y \cap U$ and $v_1 \in Y \cap V$. Since $u_1 \in \varphi^{-1}(t_0) \cap \mathrm{cl}_X(\varphi^{-1}((t_0, 1]))$, we can choose a point $u_2 \in (U - (\varphi^{-1}([t_0 + \frac{1}{n}, 1])) \cap \varphi^{-1}((t_0, 1]$, so $u_2 \in U \cap \varphi^{-1}((t_0, t_0 + \frac{1}{n}))$. Similarly, we can choose a point $v_2 \in V \cap \varphi^{-1}((t_0, t_0 + \frac{1}{n}))$. Let $A_2 = \varphi^{-1}(\varphi(u_2)\varphi(v_2))$, where $\varphi(u_2)\varphi(v_2)$ is the subinterval of the real line joining the points $\varphi(u_2)$ and $\varphi(v_2)$. Since φ is monotone, A_2 is a subcontinuum of X such that $A_2 \subset \varphi^{-1}((t_0, t_0 + \frac{1}{n}]) \subset UV$, $A_2 \cap U \neq \emptyset$ and $A_2 \cap V \neq \emptyset$. Let $m > n$ be such that $\varphi^{-1}((t_0, t_0 + \frac{1}{m}]) \cap A_2 = \emptyset$. Proceeding as before, there exists a subcontinuum

A_3 of X such that $A_3 \subset \varphi^{-1}((t_0, t_0 + \frac{1}{m}]) \subset UV$, $A_3 \cap U \neq \emptyset$ and $A_3 \cap V \neq \emptyset$. Thus $A_3 \cap A_2 = \emptyset$. Similarly, there exists a subcontinuum A_4 of X such that $A_4 \subset \varphi^{-1}((t_0, t_0 + \frac{1}{n}]) - (A_2 \cup A_3)$ such that $A_4 \cap U \neq \emptyset$ and $A_4 \cap V \neq \emptyset$. For each $i \in \{2, 3, 4\}$, fix points $r_i \in A_i \cap U$ and $s_i \in A_i \cap V$.

Let K be the convex hull of the set $\varphi(A_2) \cup \varphi(A_3) \cup \varphi(A_4) \subset (t_0, t_0 + \frac{1}{n}]$. Then $\varphi^{-1}(K)$ is a subcontinuum of UV.

Let $\zeta > 0$ be such that $2\zeta < \min\{\eta, d(\{r_2, r_3, r_4\}, X - U), d(\{s_2, s_3, s_4\}, X - V), d(\varphi^{-1}(K), X - UV)\}$ and $2\zeta < d(A_i, A_j)$, if $i \neq j$.

Take a separated ζ-chain $\mathcal{V}$ (and then $\mathcal{V}$ is a separated η-chain). We will obtain a contradiction by proving that $\mathcal{V}$ folds from V to U and $\mathcal{V}$ folds from U to V.

Let $V_1, W_1, V_2, V_3, V_4, W_2, W_3, W_4 \in \mathcal{V}$ be such that: V_1 and W_1 bound $\varphi^{-1}(K)$, V_2 and W_2 bound A_2, V_3 and W_3 bound A_3 and V_4 and W_4 bound A_4. By the choice of ζ the sets V_2W_2, V_3W_3 and V_4W_4 are pairwise disjoint. We may assume that $V_1 \leq V_2 \leq W_2 < V_3 \leq W_3 < V_4 \leq W_4 \leq W_1$. Let $R_2, R_3, R_4, S_2, S_3, S_4 \in \mathcal{V}$ be such that, for each $i \in \{2, 3, 4\}$, $r_i \in R_i$, $s_i \in S_i$ and $V_i \leq R_i, S_i \leq W_i$. By the choice of ζ, $R_2 \cup R_3 \cup R_4 \subset U$, $S_2 \cup S_3 \cup S_4 \subset V$ and $V_1W_1 \subset UV$.

Since $R_2 < S_3 < R_4$ and $S_2 < R_3 < S_4$, we obtain that $\mathcal{V}$ folds from V to U and $\mathcal{V}$ folds from U to V. This contradiction completes the proof of the theorem. ■

The other case

Proof of Theorem 1. As usual, let D be the metric in $X \times X$ given by $D((u, v), (x, y)) = \frac{1}{2}(d(u, x) + d(v, y))$, where d is a metric for X. Suppose that $\mu : X \times X \to X$ is a mean. By Theorem 10, we may assume that there exists a nondegenerate subcontinuum Y of X such that Y is not the union of two of its proper subcontinua (Y is indecomposable). We are going to find a contradiction by constructing a function $h : \{1, 2, \ldots, 4N\} \to X$ (where N is a positive integer) with the property that diameter$(\operatorname{Im} h) \geq \frac{3}{4}$diameter$(Y)$ and diameter$(\operatorname{Im} h) \leq \frac{1}{2}$diameter$(Y)$.

Claim 1. If $\mathcal{U}$ is a separated ($\frac{1}{3}$diameter(Y))-chain in X and U and V are the elements of $\mathcal{U}$ which bound Y, then there exists $\delta_1 > 0$ such that each separated δ_1-chain $\mathcal{V}$ satisfies that $\mathcal{V}$ ultrarefines $\mathcal{U}$ and $\mathcal{V}$ makes a zigzag between U and V with elements $P, Q, R, S \in \mathcal{V}$ such that $U_0 \leq P < Q < R < S \leq V_0$, where U_0 and V_0 are the elements in $\mathcal{V}$ which bound Y.

In order to prove Claim 1, let $\mathcal{U}$ be a separated $(\frac{1}{3}\text{diameter}(Y))$-chain in X and let U and V be the elements of $\mathcal{U}$ which bounds Y. Fix points $p \in Y \cap U$ and $q \in Y \cap V$. Let K_1, K_2, K_3 and K_4 be four pairwise different composants of Y ([13, Theorem 11.15, p. 203]). Since each K_i is dense in Y ([13, 5.20 (a), p. 83]), we can choose points $p_i \in K_i \cap U$ and $q_i \in K_i \cap V$. Then there exist proper subcontinua $A_1 \subset K_1$, $A_2 \subset K_2$, $A_3 \subset K_3$ and $A_4 \subset K_4$ of Y such that $p_i, q_i \in A_i$ for each $i \in \{1,2,3,4\}$. Thus A_1, A_2, A_3 and A_4 are pairwise disjoint. Let $\delta_1 > 0$ be such that $\delta_1 < \min(\{d(A_i, A_j) : i, j \in \{1,2,3,4\}$ and $i \neq j\} \cup \{d(Y, X - UV), d(\{p_1, p_2, p_3, p_4\}, X - U), d(\{q_1, q_2, q_3, q_4\}, X - V)\}$ and each separated δ_1-chain ultrarefines $\mathcal{U}$.

Let $\mathcal{V}$ be a separated δ_1-chain. Then $\mathcal{V}$ ultrarefines $\mathcal{U}$. We show that $\mathcal{V}$ makes a zigzag between V and U. Let $U_0, U_1, U_2, U_3, U_4, V_0, V_1, V_2, V_3, V_4 \in \mathcal{V}$ be such that U_0 and V_0 bounds Y and U_i and V_i bounds A_i for each $i \in \{1,2,3,4\}$. By the choice of δ_1, it follows that $U_1V_1 \cup U_2V_2 \cup U_3V_3 \cup U_4V_4 \subset U_0V_0 \subset UV$ and U_1V_1, U_2V_2, U_3V_3 and U_4V_4 are pairwise disjoint. Thus we may assume that $U_0 \leq U_1 \leq V_1 < U_2 \leq V_2 < U_3 \leq V_3 < U_4 \leq V_4 \leq V_0$. For each $i \in \{1,2,3,4\}$, let $S_i, T_i \in \mathcal{V}$ be such that $p_i \in S_i$, $q_i \in T_i$, $U_i \leq S_i, T_i \leq V_i$. By the choice of δ_1, $S_1 \cup S_2 \cup S_3 \cup S_4 \subset U$ and $T_1 \cup T_2 \cup T_3 \cup T_4 \subset V$. Since $U_0 \leq S_1 < T_2 < S_3 < T_4 \leq V_0$, we have that $\mathcal{V}$ makes a zigzag between U and V. This completes the proof of Claim 1.

Let $\delta > 0$ be such that $\delta < \frac{1}{16}(\text{diameter}(Y))$ and,

$$\text{if } D((u,v),(x,y)) < 4\delta, \text{ then } d(\mu(u,v), \mu(x,y)) < \tfrac{1}{16}(\text{diameter}(Y)).$$

Fix a separated δ-chain $\mathcal{U}$.
Let $U, V \in \mathcal{U}$ be such that U and V bound Y.

If $\text{cl}_X(U) \cap \text{cl}_X(V) \neq \emptyset$, then $U \cap V \neq \emptyset$ ($\mathcal{U}$ is a separated chain). Thus $\text{diameter}(UV) < 2\delta$. This is a contradiction since $Y \subset UV$. This proves that $\text{cl}_X(U) \cap \text{cl}_X(V) = \emptyset$.

By Lemma 7 (a), there exists an onto δ-mapping $f : \text{cl}_X(UV) \to [0,1]$ such that $\text{cl}_X(U) = f^{-1}(0)$ and $\text{cl}_X(V) = f^{-1}(1)$.

Let $\eta > 0$ be such that, if $x, y \in \text{cl}_X(UV)$ and $|f(x) - f(y)| < 2\eta$, then $d(x,y) < \delta$.

Let $\delta_1 > 0$ be as in Claim 1 applied to $\mathcal{U}$, U and V. We may assume that $\delta_1 < d(Y, X - UV)$, $\delta_1 < \delta$ and δ_1 has the property that,

$$\text{if } x, y \in \text{cl}_X(UV) \text{ and } d(x,y) < 2\delta_1, \text{ then } |f(x) - f(y)| < \eta.$$

Fix a separated δ_1-chain $\mathcal{V}$.

By the choice of δ_1, $\mathcal{V}$ ultrarefines $\mathcal{U}$ and $\mathcal{V}$ makes a zigzag between U and V with elements $P, Q, R, S \in \mathcal{V}$ such that $U_0 \leq P < Q < R < S \leq V_0$, where U_0 and V_0 are the elements in $\mathcal{V}$ which bound Y. Then $PS \subset UV$, $P \cup R \subset U$ and $Q \cup S \subset V$. Since each element $T \in \mathcal{V}$ such that $U_0 \leq T \leq V_0$ intersects Y, by the choice of δ_1, we conclude that $T \subset UV$. Thus $U_0V_0 \subset UV$. We can assume that P is the first element in the set $\{T \in \mathcal{V} : U_0 \leq T \leq V_0\}$ such that $P \subset U$; we also assume that Q is the first element in the set $\{T \in \mathcal{V} : P \leq T \leq V_0\}$ such that $Q \subset V$; R is the first element in the set $\{T \in \mathcal{V} : Q \leq T \leq V_0\}$ such that $R \subset U$ and S is the first element in the set $\{T \in \mathcal{V} : R \leq T \leq V_0\}$ such that $S \subset U$.

If $U_0 \cap V_0 \neq \emptyset$, then diameter$(Y) \leq$ diameter$(U_0V_0) < 2\delta_1 < \frac{1}{2}$diameter$(Y)$, a contradiction. Hence $U_0 \cap V_0 = \emptyset$. Thus $\mathcal{V}$ has at least three elements, so $t(\mathcal{V})$ is well defined. Since $\mathcal{V}$ is separated, $\mathrm{cl}_X(U_0) \cap \mathrm{cl}_X(V_0) = \emptyset$.

Claim 2. $\mathcal{D}(U_0, V_0) \cap ((\mathrm{cl}_X(U_0V_0) \times \mathrm{cl}_X(V_0)) \cup (\mathrm{cl}_X(V_0) \times \mathrm{cl}_X(U_0V_0)) \neq \emptyset$.

To prove Claim 2, suppose, to the contrary that $\mathcal{D}(U_0, V_0) \cap ((\mathrm{cl}_X(U_0V_0) \times \mathrm{cl}_X(V_0)) \cup (\mathrm{cl}_X(V_0) \times \mathrm{cl}_X(U_0V_0)) = \emptyset$. By Theorem 8, there exists $\eta_1 > 0$ such that if $\mathcal{V}_0$ is a separated η_1-chain in X, then $\mathcal{V}_0$ ultrarefines $\mathcal{V}$ and $\mathcal{V}_0$ does not fold from V_0 to U_0. Let $\delta_2 > 0$ be as in Claim 1 applied to $\mathcal{V}$, U_0 and V_0, we may also ask that $\delta_2 < \eta_1$. Let $\mathcal{V}_0$ be a separated δ_2-chain in X. By the choice of η_1, $\mathcal{V}_0$ ultrarefines $\mathcal{V}$ and $\mathcal{V}_0$ does not fold from V_0 to U_0. On the other hand, by the choice of δ_2, $\mathcal{V}_0$ makes a zigzag between U_0 and V_0 and then $\mathcal{V}_0$ folds from V_0 to U_0, a contradiction.
This completes the proof of Claim 2.

By Claim 2, there is a component E of $(\mathrm{cl}_X(U_0V_0) \times \mathrm{cl}_X(U_0V_0)) \cap \mu^{-1}(\mathrm{cl}_X(U_0))$ such that $E \cap \mathfrak{D}(\mathrm{cl}_X(U_0)) \neq \emptyset$ and $E \cap ((\mathrm{cl}_X(U_0V_0) \times \mathrm{cl}_X(V_0)) \cup (\mathrm{cl}_X(V_0) \times \mathrm{cl}_X(U_0V_0)) \neq \emptyset$.

We only consider the case $E \cap (\mathrm{cl}_X(V_0) \times \mathrm{cl}_X(U_0V_0)) \neq \emptyset$, the other one is analogous.

Let $\pi_1, \pi_2 : X \times X \to X$ be the respective projections on the first and second coordinates. Let $C_0 = \pi_1(E) \cup \pi_2(E)$.

Fix an element $(u_0, u_0) \in E \cap \mathfrak{D}(\mathrm{cl}_X(U_0))$. Then $u_0 \in \pi_1(E) \cap \pi_2(E)$, so C_0 is a subcontinuum of X such that $C_0 \cap \mathrm{cl}_X(U_0) \neq \emptyset$, $C_0 \cap \mathrm{cl}_X(V_0) \neq \emptyset$ and $C_0 \subset \mathrm{cl}_X(U_0V_0)$. Fix an element $(v_0, z_0) \in E \cap (\mathrm{cl}_X(V_0) \times \mathrm{cl}_X(U_0V_0))$. Then $u_0 \in C_0 \cap \mathrm{cl}_X(U_0)$ and $v_0 \in C_0 \cap \mathrm{cl}_X(V_0)$.

Let $\delta_3 > 0$ be such that $\delta_3 < \min\{\delta_1, t(\mathcal{V})\}$ and δ_3 has the property that,

$$\text{if } D((u,v),(x,y)) < \delta_3, \text{ then } d(\mu(u,v),\mu(x,y)) < t(\mathcal{V}).$$

Fix a separated δ_3-chain $\mathcal{W}$ such that $\mathcal{W}$ ultrarefines $\mathcal{V}$.

Let $U_1, V_1 \in \mathcal{W}$ be such that U_1 and V_1 bound C_0. For each $W \in \mathcal{W}$, fix a point $p_W \in W - (\bigcup\{\mathrm{cl}_X(S) : S \in \mathcal{W} - \{W\}\})$. Given a point $(x,y) \in E$, $x, y \in C_0$, so there exist $S, T \in \mathcal{W}$ such that $(x,y) \in S \times T$ and $U_1 \leq S, T \leq V_1$. We have shown that the family $\mathcal{F} = \{S \times T : S, T \in \mathcal{W}$ and $U_1 \leq S, T \leq V_1\}$ is an open cover of E. Since E is connected, there exists $n \in \mathbb{N}$ and $S_1, \ldots, S_n, T_1, \ldots, T_n \in \mathcal{W}$ such that $U_1 \leq S_1, \ldots, S_n, T_1, \ldots, T_n \leq V_1$, $(u_0, u_0) \in (S_1 \times T_1) \cap E$, $(v_0, z_0) \in (S_n \times T_n) \cap E$ and, for each $i \in 1, \ldots, n-1$, $((S_i \times T_i) \cap E) \cap ((S_{i+1} \times T_{i+1}) \cap E) \neq \emptyset$.

For each $i \in \{1, \ldots, n-1\}$, fix a pair $(\alpha(i), \beta(i)) \in (S_i \times T_i) \cap (S_{i+1} \times T_{i+1}) \cap E$. Hence $\mu(\alpha(i), \beta(i)) \in \mathrm{cl}_X(U_0)$. Put $\alpha(0) = u_0 = \beta(0) \in S_1 \cap T_1$.

Claim 3. There exists $j \in \{0, 1, \ldots, n-1\}$ and there exists an element $x \in \{\alpha(j), \beta(j)\}$ such that $x \in R$ and R is the only element of $\mathcal{V}$ containing x in its closure.

We prove Claim 3. Since $S_1 \cap T_1 \neq \emptyset$, $S_i \cap S_{i+1} \neq \emptyset$ and $T_i \cap T_{i+1} \neq \emptyset$ for each $i \in \{1, \ldots, n-1\}$, the set $\{S_1, \ldots, S_n, T_1, \ldots, T_n\}$ can be reordered as a subchain $\mathcal{W}_0$ of $\mathcal{W}$. Since $u_0 \in S_1 \cap \mathrm{cl}_X(U_0)$, $S_1 \cap U_0 \neq \emptyset$. Since $v_0 \in S_n \cap \mathrm{cl}_X(V_0)$, $S_n \cap V_0 \neq \emptyset$. By Lemma 7 (a), since $U_0 < R < V_0$, there exists an element $R_0 \in \mathcal{W}_0$ such that $R_0 \cap R \neq \emptyset$, and the only element of $\mathcal{V}$ which intersects R_0 is R, so the only element of $\mathcal{V}$ whose closure interesects R_0 is R. Thus $R_0 \subset R$. Since R_0 contains either one element of the form $\alpha(j)$ or one element of the form $\beta(j)$, we conclude that there exists $j \in \{0, 1, \ldots, n-1\}$ and there exists an element $x \in \{\alpha(j), \beta(j)\}$ such that $x \in R$ and R is the only element of $\mathcal{V}$ containing x in its closure. This ends the proof of Claim 3.

Let $j_R = \min\{j \in \{0, 1, \ldots, n-1\}$: there exists an element $x \in \{\alpha(j), \beta(j)\}$ such that $x \in R$ and R is the only element of $\mathcal{V}$ containing $x\}$. By symmetry, we may assume that $\beta(j_R) \in R$ and R is the only element of $\mathcal{V}$ containing $\beta(j_R)$ in its closure. Since $\beta(0) = u_0 \in \mathrm{cl}_X(U_0)$ and $U_0 < R$, $0 < j_R$.

Let $\mathcal{W}_1 = \{W \in \mathcal{V} : U_0 \leq W \leq R\}$. Since $\mathcal{W}_1$ is a subchain of $\mathcal{V}$, we can put $\mathcal{W}_1 = \{W_0, \ldots, W_m\}$, where $U_0 = W_0 < \ldots < W_m = R$. Note that $P, Q \in \mathcal{W}_1$. So, $Q = W_{i_0}$ for some $i_0 \in \{0, 1, \ldots, m\}$. Since $U_0 \leq P < Q < R$ and $P \cap Q = \emptyset$ ($P \subset U$ and $Q \subset V$), $1 < i_0 < m$.

Applying Lemma 7 (a), considering that the family $\{T_1, \ldots, T_{j_R}\}$ can be put as a subchain of $\mathcal{W}$, it can be shown that for each $i \in \{1, \ldots, m-1\}$, there exists $j_i \in \{1, \ldots, j_R\}$ such that:

$$\beta(j_i) \in W_i \text{ and } W_i \text{ is the only element of } \mathcal{V} \text{ containing } \beta(j_i) \text{ in its closure.}$$

We may assume that j_i is the last element in $\{1, \ldots, j_R\}$ with the described properties. Define $j_m = j_R$.

Since $E \subset \mathrm{cl}_X(U_0V_0)\times \mathrm{cl}_X(U_0V_0)$, $C_0 \subset \mathrm{cl}_X(U_0V_0)$.

Claim 4. $\alpha(i) \in \mathrm{cl}_X(U_0) \cup W_1 \cup \ldots \cup W_m$ for each $i \in \{0, 1, \ldots, j_R\}$.

In order to prove Claim 4, suppose to the contrary that there exists $i \in \{0, 1, \ldots, j_R\}$ such that $\alpha(i) \notin \mathrm{cl}_X(U_0) \cup W_1 \cup \ldots \cup W_m$. Since $\alpha(0) = u_0 \subset \mathrm{cl}_X(U_0)$, $0 < i$. Let $W \in \mathcal{V}$, be such that $\alpha(i) \in W$. If $W_m < W$, since the family $\{S_1, \ldots, S_i\}$ can be put as a subchain of $\mathcal{W}$, applying Lemma 7 (a), there exists $k \in \{1, \ldots, i-1\}$ such that $\alpha(k) \in W_m = R$ and R is the only element of $\mathcal{V}$ containing $\alpha(k)$. This contradicts the choice of j_R and proves that $W \leq W_m$, thus $W < W_0 = U_0$. Since $\alpha(i) \in C_0 \subset \mathrm{cl}_X(U_0V_0) = \bigcup\{\mathrm{cl}_X(T) : U_0 \leq T \leq V_0\}$. Since $\mathcal{V}$ is a separated chain, the only element in the family $\{\mathrm{cl}_X(T) : U_0 \leq T \leq V_0\}$ that can be intersected by W is $\mathrm{cl}_X(U_0)$. Thus $\alpha(i) \in \mathrm{cl}_X(U_0)$, a contradiction. This completes the proof of Claim 4.

In a similar way it can be proved the following claim.

Claim 5. $\beta(i) \in \mathrm{cl}_X(U_0) \cup W_1 \cup \ldots \cup W_m$ for each $i \in \{0, 1, \ldots, j_R\}$.

Define $\sigma : \{0, 1, \ldots, 2j_R - j_{i_0}\} \to \{0, 1, \ldots, 2m - i_0\}$ by

$$\sigma(i) = \begin{cases} 0, & \text{if } 0 \leq i \leq j_R \text{ and } \beta(i) \in \mathrm{cl}_X(U_0), \\ \min\{k \in \{1, \ldots, m\} : \beta(i) \in W_k\}, & \text{if } 0 \leq i \leq j_R \text{ and } \beta(i) \notin \mathrm{cl}_X(U_0), \\ 2m - \sigma(2j_R - i), & \text{if } j_R < i \leq 2j_R - j_{i_0}. \end{cases}$$

Since $\beta(0) = u_0 \in \mathrm{cl}_X(U_0)$, $\sigma(0) = 0$. Given $i \in \{1, \ldots, m\}$, $j_i \in \{0, 1, \ldots, j_R\}$, $\beta(j_i) \in W_i$ and W_i is the only element of $\mathcal{V}$ containing $\beta(j_i)$ in its closure, in particular, $\beta(j_i) \notin \mathrm{cl}_X(U_0)$. Thus $\sigma(j_i) = i$. In particular, $\sigma(j_{i_0}) = i_0$. Hence $\sigma(2j_R - j_{i_0}) = 2m - \sigma(j_{i_0}) = 2m - i_0$.

Let $i \in \{j_{i_0}, \ldots, j_R\}$, we are going to show that $i_0 \leq \sigma(i) \leq m$. Since $\sigma(j_{i_0}) = i_0$, we may assume that $j_{i_0} < i$. Suppose to the contrary that $\sigma(i) < i_0$. Note that $\beta(i) \in \mathrm{cl}_X(U_0)$ or $\beta(i) \in W_{\sigma(i)}$. In the first case, let $W \in \mathcal{V}$ be such that $\beta(i) \in W$. Since $W \cap U_0 \neq \emptyset$ and $1 < i_0$, $W < W_{i_0} = Q$. Thus, in both cases, there exists $W \in \mathcal{V}$ such that $\beta(i) \in W$ and $W < W_{i_0} = Q < R$. Consider the family $\{T_i, \ldots, T_{j_R}\} \subset \mathcal{W}$. Since $T_i \cap W \neq \emptyset$ and $\beta(j_R) \in T_{j_R} \cap W_m$ and $\{T_i, \ldots, T_{j_R}\}$ can be rearrenged as a subchain of $\mathcal{W}$, by Lemma 7 (a), there exists $l \in \{i, \ldots, j_R\}$ such that Q is the only element of $\mathcal{V}$ containing $\beta(l)$ in its closure. This contradicts the maximality of j_{i_0} and completes the proof that $i_0 \leq \sigma(i) \leq m$.

Given $i \in \{j_R, \ldots, 2j_R - j_{i_0}\}$, $j_{i_0} \leq 2j_R - i \leq j_R$. Since we have shown that $i_0 \leq \sigma(2j_R - i) \leq m$, we conclude that $m \leq 2m - \sigma(j_R - i) \leq 2m - i_0$. This proves that $\sigma(i) \in \{0, 1, \ldots, 2m - i_0\}$ for each $i \in \{0, 1, \ldots, 2j_R - j_{i_0}\}$.

Claim 6. For each $i \in \{1, 2, \dots, 2j_R - j_{i_0}\}$, $|\sigma(i) - \sigma(i-1)| \leq 1$.

To prove Claim 6, let $i, j \in \{0, 1, \dots, 2j_R - j_{i_0}\}$ be such that $|i - j| = 1$. By the choice of $\beta(i)$ and $\beta(j)$, there exists $T \in \mathcal{W}$ such that $\beta(i), \beta(j) \in T$. By the choice of δ_3 and $\mathcal{W}$, $d(\beta(i), \beta(j)) < t(\mathcal{V})$. Since $W_{k_1}, W_{k_2} \in \mathcal{V}$, if $\beta(i) \in \mathrm{cl}_X(W_{k_1})$ and $\beta(j) \in \mathrm{cl}_X(W_{k_2})$ for some $k_1, k_2 \in \{0, 1, \dots, m\}$, then $|k_1 - k_2| \leq 1$.

We consider three cases.

Case 1. $0 \leq i, j \leq j_R$.
If $\beta(i) \in \mathrm{cl}_X(U_0)$, then $\beta(j) \in \mathrm{cl}_X(U_0)$ or $\beta(j) \in W_1 -\ \mathrm{cl}_X(U_0)$, in both cases, $|\sigma(i) - \sigma(j)| \leq 1$.
If $\beta(j) \in \mathrm{cl}_X(U_0)$, similarly, $|\sigma(i) - \sigma(j)| \leq 1$.
If $\beta(i) \notin \mathrm{cl}_X(U_0)$ and $\beta(j) \notin \mathrm{cl}_X(U_0)$, then $\beta(i) \in W_{\sigma(i)}$ and $\beta(j) \in W_{\sigma(j)}$. Thus $|\sigma(i) - \sigma(j)| \leq 1$.

Case 2. $j_R < i, j \leq 2j_R - j_{i_0}$.
In this case, $1 \leq j_{i_0} \leq 2j_R - i, 2j_R - j < j_R$ and $|2j_R - i - (2j_R - j)| = 1$. Applying the first case, $|\sigma(2j_R - i) - \sigma(2j_R - j)| \leq 1$. Hence $|\sigma(i) - \sigma(j)| \leq 1$.

Case 3. $i = j_R$ and $j = j_R + 1$.
In this case, $\sigma(i) = m$ and $\sigma(j) = 2m - \sigma(j_R - 1)$. By the first case, $|\sigma(j_R) - \sigma(j_R - 1)| \leq 1$. Thus $|\sigma(i) - \sigma(j)| \leq 1$.
Therefore, Claim 6 is proved.

Since $\alpha(j_{i_0}) \in \mathrm{cl}_X(U_0) \cup W_1 \cup \ldots \cup W_m$, we can define $m_0 \in \{0, \dots, m\}$ as: $m_0 = 0$, if $\alpha(j_{i_0}) \in \mathrm{cl}_X(U_0)$, and $m_0 = \min\{j \in \{1, \dots, m\} : \alpha(j_{i_0}) \in W_j\}$, if $\alpha(j_{i_0}) \notin \mathrm{cl}_X(U_0)$.

$$\text{Let } m_1 = 2j_R - j_{i_0} + 1 + 2m - i_0 - m_0.$$

Define $\rho : \{0, 1, \dots, m_1\} \to \{0, 1, \dots, 2m - i_0\}$ by:

$$\rho(i) = \begin{cases} 0, & \text{if } 0 \leq i \leq j_R \text{ and } \alpha(i) \in \mathrm{cl}_X(U_0), \\ \min\{k \in \{1, \dots, m\} : \alpha(i) \in W_k\}, & \text{if } 0 \leq i \leq j_R \text{ and } \alpha(i) \notin \mathrm{cl}_X(U_0), \\ \rho(2j_R - i), & \text{if } j_R < i \leq 2j_R - j_{i_0}, \\ m_0 + i - (2j_R - j_{i_0} + 1), & \text{if } 2j_R - j_{i_0} < i \leq m_1. \end{cases}$$

Since $\alpha(0) = u_0 \in \mathrm{cl}_X(U_0)$, $\rho(0) = 0$. Note that $\rho(m_1) = 2m - i_0$ and $\rho(i) \leq m$ for each $i \leq 2j_R - j_{i_0}$.

Claim 7. For each $i \in \{1, \dots m_1\}$, $|\rho(i) - \rho(i-1)| \leq 1$.

To prove Claim 7, let $i, j \in \{0, 1, \dots, m_1\}$ be such that $|i - j| = 1$. We consider five cases.

Case 1. $2j_R - j_{i_0} < i, j \leq m_1$.
In this case, $|\rho(i) - \rho(j)| = |i - j| = 1$.

Case 2. $i = 2j_R - j_{i_0}$ and $j = 2j_R - j_{i_0} + 1$.
In this case, $\rho(i) = \rho(2j_R - j_{i_0}) = \rho(j_{i_0}) = m_0 = \rho(j)$.

Case 3. $0 \leq i, j \leq j_R$.
By the choice of $\alpha(i)$ and $\alpha(j)$, there exists $T \in \mathcal{W}$ such that $\alpha(i), \alpha(j) \in T$. By the choice of δ_3 and $\mathcal{W}$, $d(\alpha(i), \alpha(j)) < t(\mathcal{V})$. Thus, if $\alpha(i) \in \mathrm{cl}_X(W_{k_1})$ and $\alpha(j) \in \mathrm{cl}_X(W_{k_2})$ for some $k_1, k_2 \in \{0, 1, \ldots, m\}$, then $|k_1 - k_2| \leq 1$. Thus:

If $\alpha(i) \in \mathrm{cl}_X(U_0)$, then $\alpha(j) \in \mathrm{cl}_X(U_0)$ or $\alpha(j) \in W_1 - \mathrm{cl}_X(U_0)$, in both cases, $|\rho(i) - \rho(j)| \leq 1$.
If $\alpha(j) \in \mathrm{cl}_X(U_0)$, similarly, $|\rho(i) - \rho(j)| \leq 1$.
If $\alpha(i) \notin \mathrm{cl}_X(U_0)$ and $\alpha(j) \notin \mathrm{cl}_X(U_0)$, then $\alpha(i) \in W_{\rho(i)}$ and $\alpha(j) \in W_{\rho(j)}$. Thus $|\rho(i) - \rho(j)| \leq 1$.

Case 4. $j_R < i, j \leq 2j_R - j_{i_0}$.
In this case, $1 \leq j_{i_0} \leq 2j_R - i, 2j_R - j < j_R$. Applying Case 3, we obtain that $|\rho(2j_R - i) - \rho(2j_R - j)| \leq 1$. Hence $|\rho(i) - \rho(j)| \leq 1$.

Case 5. $i = j_R$ and $j = j_R + 1$.
In this case, $\rho(j) = \rho(j_R - 1)$. By Case 3.1, $|\rho(i) - \rho(j)| \leq 1$.
Therefore, Claim 7 is proved.

Notice that $\mathrm{cl}_X(U_0) \cup W_1 \cup \ldots \cup W_m \subset \mathrm{cl}_X(U_0V_0) \subset \mathrm{cl}_X(UV)$. Let $J = \{\alpha(i) : i \in \{0, 1, \ldots, j_R\}\} \cup \{\beta(i) : i \in \{0, 1, \ldots, j_R\}\}$.

Define $g : J \to [0, 1]$ by

$$g(\alpha(i)) = \begin{cases} f(u_0), & \text{if } \alpha(i) \in \mathrm{cl}_X(U_0), \\ f(\beta(j_{\rho(i)})), & \text{if } \alpha(i) \notin \mathrm{cl}_X(U_0), \end{cases} \text{ and}$$
$$g(\beta(i)) = \begin{cases} f(u_0), & \text{if } \beta(i) \in \mathrm{cl}_X(U_0), \\ f(\beta(j_{\sigma(i)})), & \text{if } \beta(i) \notin \mathrm{cl}_X(U_0). \end{cases}$$

Note that, if $i \in \{0, 1, \ldots, j_R\}$ and $\alpha(i) \notin \mathrm{cl}_X(U_0)$, then $\rho(i) \in \{1, \ldots, m\}$, so $j_{\rho(i)} \in \{0, 1, \ldots, j_R\}$, then $\beta(j_{\rho(i)}) \in \mathrm{cl}_X(U_0V_0)$. Thus $f(\beta(j_{\rho(i)}))$ and $g(\alpha(i))$ are well defined. Similarly, $g(\beta(i))$ is well defined for each $i \in \{0, 1, \ldots, j_R\}$.

Given $i \in \{1, \ldots, m\}$, $\beta(j_i) \in W_i - (\mathrm{cl}_X(U_0) \cup \ldots \cup W_{i-1})$, so $\sigma(j_i) = i$. Thus $g(\beta(j_i)) = f(\beta(j_i))$.

Let $\Gamma : [0, 1] \to [0, 1]$ be the PL mapping defined by the following conditions: $\Gamma(0) = g(\beta(0))$, $\Gamma(\frac{1}{2m - i_0}) = g(\beta(j_1))$, $\ldots$, $\Gamma(\frac{m}{2m - i_0}) = g(\beta(j_m))$, $\Gamma(\frac{m+1}{2m - i_0}) = g(\beta(j_{m-1}))$, $\Gamma(\frac{m+2}{2m - i_0}) = g(\beta(j_{m-2}))$, $\ldots$, $\Gamma(\frac{m+(m-i_0)}{2m - i_0}) = g(\beta(j_{i_0}))$

Let $\Phi : [0,1] \to [0,1]$ be the PL mapping defined by the following conditions: $\Phi(0) = g(\beta(0))$, $\Phi(\frac{1}{2j_R - j_{i_0}}) = g(\beta(1))$, …, $\Phi(\frac{j_R}{2j_R - j_{i_0}}) = g(\beta(j_R))$, $\Phi(\frac{j_R+1}{2j_R - j_{i_0}}) = g(\beta(j_R - 1))$, $\Phi(\frac{j_R+2}{2j_R - j_{i_0}}) = g(\beta(j_R - 2))$, …, $\Phi(\frac{j_R+(j_R-j_{i_0})}{2j_R - j_{i_0}}) = g(\beta(j_{i_0}))$.

Let $\Psi : [0,1] \to [0,1]$ be the PL mapping defined by the following conditions: $\Psi(0) = g(\beta(0))$, $\Psi(\frac{1}{2j_R - j_{i_0}}) = g(\alpha(1))$, …, $\Psi(\frac{j_R}{2j_R - j_{i_0}}) = g(\alpha(j_R))$, $\Psi(\frac{j_R+1}{2j_R - j_{i_0}}) = g(\alpha(j_R - 1))$, $\Psi(\frac{j_R+2}{2j_R - j_{i_0}}) = g(\alpha(j_R - 2))$, …, $\Psi(\frac{j_R+(j_R-j_{i_0})}{2j_R - j_{i_0}}) = g(\alpha(j_{i_0}))$.

Let $\Delta : [0,1] \to [0,1]$ be the PL mapping defined by the following conditions: $\Delta(0) = g(\beta(j_R))$, $\Delta(\frac{1}{j_R - j_{i_0}}) = g(\beta(j_R - 1))$, …, $\Delta(\frac{j_R - j_{i_0}}{j_R - j_{i_0}}) = g(\beta(j_{i_0}))$.

Since $i_0 > 0$ and $\beta(j_{i_0}) \in Q \subset V$, so $g(\beta(j_{i_0})) = f(\beta(j_{i_0})) = 1$. Since $\beta(j_R) \in R \subset U$, $g(\beta(j_R)) = g(\beta(j_m)) = f(\beta(j_m)) = f(\beta(j_R)) = 0$. Therefore, $\Gamma(1) = 1$, $\Phi(1) = 1$, $\Delta(0) = 0$ and $\Delta(1) = 1$. Hence Δ is a jump mapping.

We want to apply Theorem 6 to the mappings Γ and Φ.

Claim 8. $\Gamma(\frac{\sigma(i)}{2m-i_0}) = \Phi(\frac{i}{2j_R - i_0})$ for each $i \in \{0, 1, \ldots, 2j_R - j_{i_0}\}$.

To prove Claim 8, we consider three cases.

Case 1. $0 \le i \le j_R$ and $\beta(i) \in \mathrm{cl}_X(U_0)$.
In this case $\sigma(i) = 0$, $\Gamma(\frac{\sigma(i)}{2m-i_0}) = g(\beta(0)) = f(u_0)$, $\Phi(\frac{i}{2j_R - i_0}) = g(\beta(i)) = f(u_0)$. Hence, $\Gamma(\frac{\sigma(i)}{2m-i_0}) = \Phi(\frac{i}{2j_R-i_0})$.

Case 2. $0 \le i \le j_R$ and $\beta(i) \notin \mathrm{cl}_X(U_0)$.
By definition of $\sigma(i)$, $\sigma(i) \in \{1, \ldots, m\}$. By definition of $g(\beta(i))$, $g(\beta(i)) = f(\beta(j_{\sigma(i)}))$. Thus $\Phi(\frac{i}{2j_R-i_0}) = f(\beta(j_{\sigma(i)}))$. Since $f(\beta(j_k)) = g(\beta(j_k))$ for each $k \in \{1, \ldots, m\}$ and $\sigma(i) \in \{1, \ldots, m\}$, we obtain that $\Gamma(\frac{\sigma(i)}{2m-i_0}) = g(\beta(j_{\sigma(i)})) = f(\beta(j_{\sigma(i)}))$. Hence, $\Gamma(\frac{\sigma(i)}{2m-i_0}) = \Phi(\frac{i}{2j_R-i_0})$.

Case 3. $j_R < i \le 2j_R - j_{i_0}$.
In this case, $1 \le j_{i_0} \le 2j_R - i < j_R$. As we proved after the definition of σ this inequalities imply that $1 < i_0 \le \sigma(2j_R - i) \le m$. Thus $g(\beta(j_{\sigma(2j_R-1)})) = f(\beta(j_{\sigma(2j_R-1)}))$ and $\sigma(i) = 2m - \sigma(2j_R - i) \in \{m, \ldots, 2m - i_0\}$. By the definition of Γ, $\Gamma(\sigma(i)) = g(\beta(j_{2m-\sigma(i)})) = g(\beta(j_{\sigma(2j_R-i)})) = f(\beta(j_{\sigma(2j_R-i)}))$. On the other hand, $\Phi(\frac{i}{2j_R-i_0}) = g(\beta(2j_R - i))$. Since $1 < \sigma(2j_R - i)$, $\beta(2j_R - i) \notin \mathrm{cl}_X(U_0)$, so $\Phi(\frac{i}{2j_R-i_0}) = g(\beta(2j_R - i)) = f(\beta(j_{\sigma(2j_R-i)})) = \Gamma(\sigma(i))$.

This completes the proof of Claim 8.

Let $\Psi_0 : [0,1] \to [0,1]$ be the PL mapping which is the common extension of the following two mappings: $\Psi(2t)$, if $t \in [0, \frac{1}{2}]$, and $\Gamma((\frac{m_0}{2m-i_0})(4-4t)+4t-3)$, if $\frac{3}{4} \le t \le 1$.

Since $i_0 < m$, $0 < 2m - i_0 - m_0$. Notice that Ψ_0 is supported by the partition $0 < \frac{1}{2(2j_R - j_{i_0})} < \ldots < \frac{2j_R - j_{i_0}}{2(2j_R - j_{i_0})} = \frac{1}{2} < \frac{3}{4} < \frac{3}{4} + \frac{1}{4}(\frac{1}{2m-i_0-m_0}) < \ldots < \frac{3}{4} + \frac{1}{4}(\frac{2m-i_0-m_0}{2m-i_0-m_0}) = 1$, which divides the interval $[0,1]$ into $m_1 = 2j_R - j_{i_0} + 1 + 2m - i_0 - m_0$ subintervals.

Claim 9. $\Psi_0(\frac{i}{2(2j_R - j_{i_0})}) = \Gamma(\frac{\rho(i)}{2m-i_0})$ for each $i \in \{0, \ldots, 2j_R - j_{i_0}\}$.

To prove Claim 9, we consider three cases.

Case 1. $0 \le i \le j_R$ and $\alpha(i) \in \mathrm{cl}_X(U_0)$.

In this case, $\rho(i) = 0$ and $\Gamma(\frac{\rho(i)}{2m-i_0)}) = g(\beta(0)) = f(u_0)$. On the other hand, $0 \le \frac{i}{2(2j_R - j_{i_0})} \le \frac{1}{2}$, so $\Psi_0(\frac{i}{2(2j_R - j_{i_0})}) = \Psi(\frac{i}{2j_R - j_{i_0}}) = g(\alpha(i)) = f(u_0)$. Hence, $\Psi_0(\frac{i}{2(2j_R - j_{i_0})}) = \Gamma(\frac{\rho(i)}{2m-i_0})$.

Case 2. $0 \le i \le j_R$ and $\alpha(i) \notin \mathrm{cl}_X(U_0)$,

By definition of $\rho(i)$, $\alpha(i) \in W_{\rho(i)} - (\mathrm{cl}_X(U_0) \cup W_1 \cup \ldots \cup W_{\rho(i)-1})$. By definition of $g(\alpha(i))$, $g(\alpha(i)) = f(\beta(j_{\rho(i)}))$. Thus $\Psi_0(\frac{i}{2(2j_R - i_0)}) = \Psi(\frac{i}{2j_R - j_{i_0}}) = g(\alpha(i)) = f(\beta(j_{\rho(i)}))$. Since $f(\beta(j_k)) = g(\beta(j_k))$ for each $k \in \{1, \ldots, m\}$ and $\rho(i) \in \{1, \ldots, m\}$, we obtain that $\Gamma(\frac{\rho(i)}{2m-i_0}) = g(\beta(j_{\rho(i)})) = f(\beta(j_{\rho(i)}))$. Hence, $\Psi_0(\frac{i}{2(2j_R - i_0)}) = \Gamma(\frac{\rho(i)}{2m-i_0})$.

Case 3. $j_R < i \le 2j_R - j_{i_0}$.

In this case, $j_{i_0} \le 2j_R - i < j_R$, so $\Psi_0(\frac{i}{2(2j_R - j_{i_0})}) = \Psi(\frac{i}{2j_R - j_{i_0}}) = g(\alpha(2j_R - i)) = \Psi(\frac{2j_R - i}{2j_R - j_{i_0}}) = \Psi_0(\frac{2j_R - i}{2(2j_R - j_{i_0})}) =$ (by the first two cases) $\Gamma(\frac{\rho(2j_R - i)}{2m-i_0}) = \Gamma(\frac{\rho(i)}{2m-i_0})$. This ends Case 3 and completes the proof of Claim 9.

It is easy to check that, if $2j_R - j_{i_0} < i \le m_1$, then $\Psi_0(\frac{3}{4} + \frac{1}{4}(\frac{i-(2j_R - j_{i_0}+1)}{2m-i_0-m_0})) = \Gamma(\frac{m_0 + i - (2j_R - j_{i_0}+1)}{2m-i_0})$.

Therefore, we can apply Theorem 6 to obtain a jump mapping Ω_0 such that $\Psi_0 = \Gamma \circ \Omega_0$.

Let $\Omega : [0,1] \to [0,1]$ be the PL mapping given by $\Omega(t) = \Omega_0(\frac{t}{2})$. Then $\Psi = \Gamma \circ \Omega$ and $\Omega(0) = 0$.

We also can apply Theorem 6 to Φ, Γ and σ and obtain a jump mapping Ξ such that $\Phi = \Gamma \circ \Xi$.

By Theorem 4, there exist a jump mapping Θ and a PL mapping Λ such that $\Lambda(1) = 1$ and $\Gamma \circ \Theta = \Delta \circ \Lambda$.

By Theorem 3, there exist jump mappings Π and Σ such that $\Theta \circ \Pi = \Xi \circ \Sigma$.

By Theorem 5, there exist a PL mapping $\digamma$ and a jump mapping Υ such that $\digamma(0) = 0$ and $\Omega \circ \Upsilon = \Theta \circ \digamma$.

By Theorem 3, there exist jump mappings ζ, κ such that $\Sigma \circ \zeta = \Upsilon \circ \kappa$.

Observe that $\Phi \circ \Sigma \circ \zeta = \Gamma \circ \Xi \circ \Sigma \circ \zeta = \Gamma \circ \Theta \circ \Pi \circ \zeta = \Delta \circ \Lambda \circ \Pi \circ \zeta$ and $\Psi \circ \Upsilon \circ \kappa = \Gamma \circ \Omega \circ \Upsilon \circ \kappa = \Gamma \circ \Theta \circ \digamma \circ \kappa = \Delta \circ \Lambda \circ \digamma \circ \kappa$.

Hence $\Phi \circ \Sigma \circ \zeta = \Delta \circ \Lambda \circ \Pi \circ \zeta$ and $\Psi \circ \Upsilon \circ \kappa = \Delta \circ \Lambda \circ \digamma \circ \kappa$.

Notice also that $\Phi(0) = (\Phi \circ \Sigma \circ \zeta)(0) = (\Delta \circ \Lambda \circ \Pi \circ \zeta)(0) = (\Delta \circ \Lambda)(0)$, $\Phi(1) = (\Phi \circ \Sigma \circ \zeta)(1) = (\Delta \circ \Lambda \circ \Pi \circ \zeta)(1) = (\Delta \circ \Lambda)(1)$ and $\Psi(0) = (\Psi \circ \Upsilon \circ \kappa)(0) = (\Delta \circ \Lambda \circ \digamma \circ \kappa)(0) = (\Delta \circ \Lambda)(0)$.

Let $0 = r_1 < r_2 < \ldots < r_N = 1$ be a partition of $[0, 1]$ such that, for each $i \in \{2, \ldots, N\}$ and each $\xi \in \{\Sigma \circ \zeta, \Lambda \circ \Pi \circ \zeta, \Upsilon \circ \kappa, \Lambda \circ \digamma \circ \kappa\}$, we have $|\xi(r_i) - \xi(r_{i-1})| < \frac{1}{2(2j_R - j_{i_0})}$.

Given a nonempty closed set $B \subset \mathbb{R}$ and a point $x \in \mathbb{R}$, choose the lowest (in the order of the real line) point $\tau(x, B) \in B$ such that $|x - \tau(x, B)| = \min\{|x - y| : y \in B\}$.

Let $\gamma : \{1, 2, \ldots, 2N\} \to [0, 1]$ be given by

$$\gamma(i) = \begin{cases} \tau(\Sigma(\zeta(r_i)), \{\frac{0}{2j_R - j_{i_0}}, \frac{1}{2j_R - j_{i_0}}, \ldots, \frac{2j_R - j_{i_0}}{2j_R - j_{i_0}}\}), & \text{if } i \in \{1, 2, \ldots, N\}, \\ \tau(\Lambda(\Pi(\zeta(r_{2N-i+1}))), \{\frac{0}{j_R - j_{i_0}}, \frac{1}{j_R - j_{i_0}}, \ldots, \frac{j_R - j_{i_0}}{j_R - j_{i_0}}\}), & \text{if } i \in \{N + 1, \ldots, 2N\}. \end{cases}$$

Let $\lambda : \{1, 2, \ldots, N\} \cup \{2N + 1, \ldots, 3N\} \to [0, 1]$ be given by

$$\lambda(i) = \begin{cases} \tau(\Upsilon(\kappa(r_i)), \{\frac{0}{2j_R - j_{i_0}}, \frac{1}{2j_R - j_{i_0}}, \ldots, \frac{2j_R - j_{i_0}}{2j_R - j_{i_0}}\}), & \text{if } i \in \{1, 2, \ldots, N\}, \\ \tau(\Lambda(\digamma(\kappa(r_{i-2N}))), \{\frac{0}{j_R - j_{i_0}}, \frac{1}{j_R - j_{i_0}}, \ldots, \frac{j_R - j_{i_0}}{j_R - j_{i_0}}\}), & \text{if } i \in \{2N + 1, \ldots, 3N\}. \end{cases}$$

Let $L = \{1, \ldots, 4N\}$ and let $F, G : L \to X$ be the functions defined by:

$$F(i) = \begin{cases} \beta(k), & \text{if } \gamma(i) = \frac{k}{2j_R - j_{i_0}}, \ k \in \{0, 1, \ldots, j_R\}, \ i \in \{1, \ldots, N\}, \\ \beta(2j_R - k), & \text{if } \gamma(i) = \frac{k}{2j_R - j_{i_0}}, \ k \in \{j_R + 1, \ldots, 2j_R - j_{i_0}\}, \ i \in \{1, \ldots, N\}, \\ \beta(j_R - k), & \text{if } \gamma(i) = \frac{k}{j_R - j_{i_0}}, \ k \in \{0, \ldots, j_R - j_{i_0}\}, \ i \in \{N + 1, \ldots, 2N\}, \\ F(i - 2N), & \text{if } i \in \{2N + 1, \ldots, 4N\}, \end{cases}$$

and

$$G(i) =$$
$$\begin{cases} \alpha(k), & \text{if } \lambda(i) = \frac{k}{2j_R - j_{i_0}},\ k \in \{0, 1, \dots, j_R\},\ i \in \{1, \dots, N\}, \\ \alpha(2j_R - k), & \text{if } \lambda(i) = \frac{k}{2j_R - j_{i_0}},\ k \in \{j_R + 1, \dots, 2j_R - j_{i_0}\},\ i \in \{1, \dots, N\}, \\ G(2N - i + 1), & \text{if } i \in \{N + 1, \dots, 2N\}, \\ \beta(j_R - k), & \text{if } \lambda(i) = \frac{k}{j_R - j_{i_0}},\ k \in \{0, \dots, j_R - j_{i_0}\},\ i \in \{2N + 1, \dots, 3N\}, \\ G(6N - i + 1), & \text{if } i \in \{3N + 1, \dots, 4N\}. \end{cases}$$

In the following claim we resume some easy to check equalities.

Claim 10. $F(1) = u_0 = G(1)$; $F(N) = \beta(j_{i_0})$; $G(N) = \alpha(j_{i_0})$; $F(N+1) = \beta(j_{i_0})$; $G(N+1) = \alpha(j_{i_0})$; $F(2N) = \beta(j_R - k_0)$, where $\gamma(2N) = \frac{k_0}{j_R - j_{i_0}} = \tau(\Lambda(0), \{\frac{0}{j_R - j_{i_0}}, \dots, \frac{j_R - j_{i_0}}{j_R - j_{i_0}}\})$; $G(2N) = u_0$; $F(2N+1) = u_0$, $G(2N+1) = \beta(j_R - k_0)$; $F(3N) = \beta(j_{i_0})$; $G(3N) = \beta(j_R - k_1)$, where $\frac{k_1}{j_R - j_{i_0}} = \lambda(3N) = \tau(\Lambda(F(\kappa(1))), \{\frac{0}{j_R - j_{i_0}}, \dots, \frac{j_R - j_{i_0}}{j_R - j_{i_0}}\})$; $F(3N+1) = \beta(j_{i_0})$; $G(3N+1) = G(3N)$; $F(4N) = \beta(j_R - k_0)$ and $G(4N) = \beta(j_R - k_0)$.

Let $h : L \to X$ be given by

$$h(i) = \mu(F(i), G(i)).$$

From Claim 10, the following claim is immediate.

Claim 11. $h(1) = u_0$; $h(N) = h(N+1)$; $h(2N) = h(2N+1)$; $h(3N) = h(3N+1)$ and $h(4N) = \beta(j_R - k_0)$, for some $k_0 \in \{0, 1, \dots, j_R - j_{i_0}\}$.

Claim 12. $d(F(i), F(i+1)) < \delta_3$ for each $i \in L - \{N, 2N, 3N, 4N\}$.

To prove Claim 12, first we consider the case that $i \in \{1, \dots, N-1\}$.
Let $\gamma(i) = \frac{l_1}{2j_R - j_{i_0}}$ and $\gamma(i+1) = \frac{l_2}{2j_R - j_{i_0}}$. By the choice of r_i and r_{i+1}, $|\Sigma(\zeta(r_i) - \Sigma(\zeta(r_{i+1})| < \frac{1}{2(2j_R - j_{i_0})}$. This implies that $|l_1 - l_2| \leq 1$. Analizing the possibilities for l_1 and l_2 ($l_1, l_2 \leq j_R$, $l_1 \leq j_R < l_2$, $l_2 \leq j_R < l_1$ and $j_R \leq l_1, l_2$) it can be seen that $F(i) = \beta(i_1)$ and $F(i+1) = \beta(i_2)$ for some $i_1, i_2 \in \{1, \dots, j_R\}$ such that $|i_1 - i_2| \leq 1$. By the choice of $\beta(i_1)$ and $\beta(i_2)$, there exists $T \in \mathcal{W}$ such that $\beta(i_1), \beta(i_2) \in T$. Therefore, $d(F(i), F(i+1)) < \delta_3$.

The case $i \in \{N+1, \dots, 2N-1\}$ is similar. The case $i \in \{2N+1, \dots, 3N-1\} \cup \{3N+1, \dots, 4N-1\}$ follows from the previous ones and the definition of F.

This completes the proof of Claim 12.

Similar arguments can be used to show the following claim.

Claim 13. $d(G(i), G(i+1)) < \delta_3$ for each $i \in L - \{N, 2N, 3N, 4N\}$.

From the choice of δ_3 and Claims 11, 12 and 13, we obtain the following.

Claim 14. $d(h(i), h(i+1)) < t(\mathcal{V})$ for each $i \in \{1, \dots, 4N-1\}$.

Claim 15. Let $P_0 \in \mathcal{V}$ be such that $P < P_0 \leq Q$. Then $h(L) \cap P_0 \neq \emptyset$.

We prove Claim 15. First we show that $h(L) \cap Q \neq \emptyset$.

By Claim 11, $h(1) = u_0 \in h(L) \cap \operatorname{cl}_X(U_0)$ and $h(4N) = \beta(j_R - k_0)$ for some $k_0 \in \{0, 1, \dots, j_R - j_{i_0}\}$. Note that $j_{i_0} \leq j_R - k_0 \leq j_R$. If $j_R - k_0 = j_{i_0}$, then $h(4N) = \beta(j_R - k_0) = \beta(j_{i_0}) \in W_{i_0} \cap h(L) = Q \cap h(L)$ and we finish. Thus, we may assume that $j_{i_0} < j_R - k_0$.

Since $\beta(j_R - k_0) \in T_{j_R - k_0}$, from Claim 5 we have that $T_{j_R - k_0} \cap U_0 \neq \emptyset$ or there exists $i_1 \in \{1, \dots, m\}$ such that $\beta(j_R - k_0) \in W_{i_1}$. We consider two cases.

Case 1. $T_{j_R - k_0} \cap U_0 \neq \emptyset$ or $\beta(j_R - k_0) \in W_{i_1}$ for some $i_1 < i_0$.
Consider the subchain $\mathcal{W}_2$ of $\mathcal{W}$ which can be constructed by ordering the elements $T_{j_R - k_0}, \dots, T_{j_R} \in \mathcal{W}$. Since $U_0 < Q < R$ and $\beta(j_R) \in T_{j_R} \cap W_R$, by Lemma 7 (a), there exists one element T_i (where $j_R - k_0 \leq i \leq j_R$) of $\mathcal{W}_2$ such that $T_i \cap Q \neq \emptyset$ and Q is the only element of $\mathcal{V}$ which intersects T_i. Thus Q is the only element of $\mathcal{V}$ whose closure intersects T_i. Hence $\beta(i) \in T_i \subset Q$ and Q is the only element of $\mathcal{V}$ which contains $\beta(i)$ in its closure. Recall that j_{i_0} was the last index with this property, we have obtained a contradiction since $j_{i_0} < i$. We have shown that this case is impossible.

Case 2. $\beta(j_R - k_0) \in W_{i_1}$ for some $i_0 \leq i_1$.
If $i_0 = i_1$, then $h(4N) = \beta(j_R - k_0) \in h(L) \cap Q$ and we finish, so we may assume that $i_0 < i_1$. Let $A = \bigcup\{\operatorname{cl}_X(T) : T \in \mathcal{V} \text{ and } T < Q\}$ and $B = \bigcup\{\operatorname{cl}_X(T) : T \in \mathcal{V} \text{ and } Q < T\}$. Notice that $X = A \cup B \cup Q$, $\operatorname{cl}_X(U_0) \subset A$ and $\operatorname{cl}_X(W_{i_1}) \subset B$. Thus $h(1) \in A$ and $h(4N) \in B$. Since $t(\mathcal{V}) < d(A, B)$, Claim 14 implies that there exists $i \in \{1, \dots, 4N\}$ such that $h(i) \notin A \cup B$. Thus $h(i) \in h(L) \cap Q$.

This completes the proof that $h(L) \cap Q \neq \emptyset$.

To finish the proof of Claim 15, let $P_0 \in \mathcal{V}$ be such that $P < P_0 < Q$. Since $U_0 \leq P$, $h(L) \cap \operatorname{cl}_X(U_0) \neq \emptyset$ and $h(L) \cap Q \neq \emptyset$, a similar argument as in Case 2 shows that $h(L) \cap P_0 \neq \emptyset$. This completes the proof of Claim 15.

Claim 16. $\frac{3}{4}\text{diameter}(Y) \leq \text{diameter}(h(L))$.

We prove Claim 16. Let $x, y \in Y$ be such that $d(x, y) = \text{diameter}(Y)$. Since $Y \subset UV$, there exist $U_x, U_y \in \mathcal{U}$ such that $x \in U_x$, $y \in U_y$ and $U \leq U_x, U_y \leq V$. We may assume that $U_x \leq U_y$.

Given $T \in \mathcal{U}$ with $U < T < V$, since $P \subset U$ and $Q \subset V$, by Lemma 7 (a), there exists $W \in \mathcal{V}$ such that the only element of $\mathcal{U}$ which intersects W is T and $P \leq W \leq Q$. Note that $W \neq P$ and $W \neq Q$. By Claim 15, $\emptyset \neq h(L) \cap W \subset h(L) \cap T$. Since $\emptyset \neq h(L) \cap Q \subset h(L) \cap V$, we conclude that, for each $T \in \mathcal{U}$ with $U < T \leq V$, $h(L) \cap T \neq \emptyset$.

If $U = U_x$, let T be the element in $\mathcal{U}$ such that $U < T < V$ and $U \cap T \neq \emptyset$ (the next element in the chain $\mathcal{U}$ after U). Thus there exists $i_x \in \{1, \ldots, 4N\}$ such that $h(i_x) \in T$. Hence $d(x, h(i_x)) \leq \text{diameter}(U_x \cup T) < 2\delta < \frac{1}{8}(\text{diameter}(Y))$. In the case $U < U_x$, $h(L) \cap U_x \neq \emptyset$. Thus there exists $i_x \in \{1, \ldots, 4N\}$ such that $h(i_x) \in U_x$. Hence $d(x, h(i_x)) \leq \text{diameter}(U_x) < \delta < \frac{1}{8}(\text{diameter}(Y))$. In any case, there exists $i_x \in \{1, \ldots, 4N\}$ such that $d(x, h(i_x)) \leq \frac{1}{8}(\text{diameter}(Y))$.

Similarly, there exists $i_y \in \{1, \ldots, 4N\}$ such that $d(y, h(i_y)) \leq \frac{1}{8}(\text{diameter}(Y))$.

Thus $\text{diameter}(Y) = d(x, y) \leq \frac{1}{4}(\text{diameter}(Y)) + d(h(i_x), h(i_y))$. Therefore, $\frac{3}{4}(\text{diameter}(Y)) \leq \text{diameter}(h(L))$. We have shown Claim 16.

We have defined, for each $i \in \{1, \ldots, m\}$, $j_i \in \{1, \ldots, j_R\}$ with the property that $\beta(j_i) \in W_i$. To extend this definition, we consider the formal symbol j_0 and we put $\beta(j_0) = u_0$. With this convention, $f(\beta(j_0)) = f(u_0)$. Since $\sigma(0) = 0 = \rho(0)$, $g(\beta(i)) = f(\beta(j_{\sigma(i)}))$ and $g(\alpha(i)) = f(\beta(j_{\rho(i)}))$ for each $i \in \{0, 1, \ldots, j_R\}$.

Claim 17. Let $r \in [0, 1]$, $k \in \{0, 1, \ldots, j_R - j_{i_0}\}$ and $l \in \{0, 1, \ldots, 2j_R - j_{i_0}\}$ have the properties that $\tau(\Lambda(\Pi(\zeta(r))), \{\frac{0}{j_R - j_{i_0}}, \ldots, \frac{j_R - j_{i_0}}{j_R - j_{i_0}}\}) = \frac{k}{j_R - j_{i_0}}$ and $\tau(\Sigma(\zeta(r)), \{\frac{0}{2j_R - j_{i_0}}, \ldots, \frac{2j_R - j_{i_0}}{2j_R - j_{i_0}}\}) = \frac{l}{2j_R - j_{i_0}}$. If $l \leq j_R$, then $d(\beta(j_R - k), \beta(l)) < 3\delta$. If $j_R < l$, then $d(\beta(j_R - k), \beta(2j_R - l)) < 3\delta$.

In order to prove Claim 17, first we show $\left| f(\beta(j_{\sigma(j_R - k)})) - \Delta(\Lambda(\Pi(\zeta(r))))\right| < \eta$.

Consider first the case that $\frac{k}{j_R - j_{i_0}} < \Lambda(\Pi(\zeta(r))$. By the definition of τ, $\frac{k}{j_R - j_{i_0}} < \Lambda(\Pi(\zeta(r)) < \frac{k+1}{j_R - j_{i_0}}$. By the definition of Δ, $\left|\Delta(\frac{k+1}{j_R - j_{i_0}}) - \Delta(\frac{k}{j_R - j_{i_0}})\right| \geq \left|\Delta(\Lambda(\Pi(\zeta(r))) - \Delta(\frac{k}{j_R - j_{i_0}})\right|$, $\Delta(\frac{k}{j_R - j_{i_0}}) = g(\beta(j_R - k)) = f(\beta(j_{\sigma(j_R - k)}))$ and $\Delta(\frac{k+1}{j_R - j_{i_0}}) = g(\beta(j_R - (k+1))) = f(\beta(j_{\sigma(j_R - (k+1))}))$. Claim 6 implies that $|\sigma(j_R - k) - \sigma(j_R - (k+1))| \leq 1$. Thus $W_{\sigma(j_R - k)} \cap W_{\sigma(j_R - (k+1))} \neq \emptyset$ this implies that $\text{diameter}(\text{cl}_X(W_{\sigma(j_R - k)} \cup W_{\sigma(j_R - (k+1))})) < 2\delta_1$. Note that $\beta(j_{\sigma(j_R - k)})$, $\beta(j_{\sigma(j_R - (k+1))}) \in \text{cl}_X(W_{\sigma(j_R - k)} \cup W_{\sigma(j_R - (k+1))})$ (even in the case that $\sigma(j_R - k)$

or $\sigma(j_R-(k+1))$ is equal to 0), so $d(\beta(j_{\sigma(j_R-k)}),\beta(j_{\sigma(j_R-(k+1))}))<2\delta_1$. By the choice of δ_1, we obtain that $\left|f(\beta(j_{\sigma(j_R-k)}))-f(\beta(j_{\sigma(j_R-(k+1))}))\right|<\eta$. Therefore $\left|f(\beta(j_{\sigma(j_R-k)}))-\Delta(\Lambda(\Pi(\zeta(r)))\right|=\left|\Delta(\Lambda(\Pi(\zeta(r)))-\Delta(\frac{k}{j_R-j_{i_0}})\right|<\eta$.

In the case that $\Lambda(\Pi(\zeta(r))<\frac{k}{j_R-j_{i_0}}$, $\frac{k-1}{j_R-j_{i_0}}<\Lambda(\Pi(\zeta(r))<\frac{k}{j_R-j_{i_0}}$. So a similar argument as in the paragraph above can be made, by changing $k+1$ by $k-1$, to obtain the desired inequality. Finally, if $\Lambda(\Pi(\zeta(r))=\frac{k}{j_R-j_{i_0}}$, then $\Delta(\Lambda(\Pi(\zeta(r)))=\Delta(\frac{k}{j_R-j_{i_0}})=f(\beta(j_{\sigma(j_R-k)}))$ and the inequality is immediate.

Now, we analize the possible cases for l. First, suppose that $l\leq j_R$.

Next, we prove that $\left|f(\beta(j_{\sigma(l)}))-\Phi(\Sigma(\zeta(r)))\right|<\eta$.

Consider first the case that $\frac{l}{2j_R-j_{i_0}}<\Sigma(\zeta(r))$. By the definition of τ, $\frac{l}{2j_R-j_{i_0}}<\Sigma(\zeta(r))<\frac{l+1}{2j_R-j_{i_0}}$. By the definition of Φ, $\left|\Phi(\frac{l}{2j_R-j_{i_0}})-\Phi(\Sigma(\zeta(r)))\right|\leq\left|\Phi(\frac{l+1}{2j_R-j_{i_0}})-\Phi(\frac{l}{2j_R-j_{i_0}})\right|$ and $\Phi(\frac{l}{2j_R-j_{i_0}})=g(\beta(l))=f(\beta(j_{\sigma(l)}))$. In the case that $l+1\leq j_R$, $\Phi(\frac{l+1}{2j_R-j_{i_0}})=g(\beta(l+1))=f(\beta(j_{\sigma(l+1)}))$. By Claim 6, $|\sigma(l)-\sigma(l+1)|\leq 1$. Thus $W_{\sigma(l)}\cap W_{\sigma(l+1)}\neq\emptyset$ and diameter$(\mathrm{cl}_X(W_{\sigma(l)}\cup W_{\sigma(l+1)}))<2\delta_1$. Since $\beta(j_{\sigma(l)}),\beta(j_{\sigma(l+1)})\in\mathrm{cl}_X(W_{\sigma(l)}\cup W_{\sigma(l+1)})$, we obtain $d(\beta(j_{\sigma(l)}),\beta(j_{\sigma(l+1)}))<2\delta_1$. By the choice of δ_1, $\left|f(\beta(j_{\sigma(l)}))-f(\beta(j_{\sigma(l+1)}))\right|<\eta$. Hence $\left|f(\beta(j_{\sigma(l)}))-\Phi(\Sigma(\zeta(r)))\right|=\left|\Phi(\Sigma(\zeta(r)))-\Phi(\frac{l}{2j_R-j_{i_0}})\right|<\eta$. In the case that $j_R<l+1$, we have that $l=j_R$, $l+1=j_R+1$, $\Phi(\frac{l}{2j_R-j_{i_0}})=g(\beta(l))=f(\beta(j_{\sigma(l)}))$ and $\Phi(\frac{l+1}{2j_R-j_{i_0}})=g(\beta(l-1))=f(\beta(j_{\sigma(l-1)}))$. Hence a similar argument as before leads to the proof that $\left|f(\beta(j_{\sigma(l)}))-\Phi(\Sigma(\zeta(r)))\right|<\eta$.

The case $\Sigma(\zeta(r))<\frac{l}{2j_R-j_{i_0}}$ is similar and the case $\Sigma(\zeta(r))=\frac{l}{2j_R-j_{i_0}}$ is immediate.

Therefore, if $l\leq j_R$, then $\left|f(\beta(j_{\sigma(l)}))-\Phi(\Sigma(\zeta(r)))\right|<\eta$.

Since $\Delta(\Lambda(\Pi(\zeta(r)))=\Phi(\Sigma(\zeta(r)))$, assuming that $l\leq j_R$, we obtain that $\left|f(\beta(j_{\sigma(j_R-k)}))-f(\beta(j_{\sigma(l)}))\right|<2\eta$. By the choice of η, $d(\beta(j_{\sigma(j_R-k)}),\beta(j_{\sigma(l)})<\delta$. Since $\beta(j_R-k),\beta(j_{\sigma(j_R-k)})\in\mathrm{cl}_X(W_{\sigma(j_R-k)})$ and $\beta(l),\beta(j_{\sigma(l)})\in\mathrm{cl}_X(W_{\sigma(l)})$, $d(\beta(j_R-k),\beta(j_{\sigma(j_R-k)})),d(\beta(l),\beta(j_{\sigma(l)}))<\delta_1<\delta$. Hence $d(\beta(j_R-k),\beta(l))<3\delta$.

In the case that $j_R<l$, similar arguments can be used to show that $d(\beta(j_R-k),\beta(2j_R-l))<3\delta$. We have finished the proof of Claim 17.

Mimicking the proof of Claim 17, the following claim can be proved.

Claim 18. Let $r \in [0,1]$, $k \in \{0, 1, \ldots, j_R - j_{i_0}\}$ and $l \in \{0, 1, \ldots, 2j_R - j_{i_0}\}$ have the property that $\tau(\Lambda(F(\kappa(r))), \{\frac{0}{j_R - j_{i_0}}, \ldots, \frac{j_R - j_{i_0}}{j_R - j_{i_0}}\}) = \frac{k}{j_R - j_{i_0}}$ and $\tau(\Upsilon(\kappa(r)), \{\frac{0}{2j_R - j_{i_0}}, \ldots, \frac{2j_R - j_{i_0}}{2j_R - j_{i_0}}\}) = \frac{l}{2j_R - j_{i_0}}$. If $l \leq j_R$, then $d(\beta(j_R - k), \alpha(l)) < 3\delta$. If $j_R < l$, then $d(\beta(j_R - k), \alpha(2j_R - l)) < 3\delta$.

Claim 19. For each $i \in \{1, \ldots, 4N\}$, $d(h(i), \mathrm{cl}_X(U_0)) < \frac{1}{16}(\text{diameter}(Y))$.

We consider four cases.

Case 1. $i \in \{1, \ldots, N\}$.

Since $\Sigma(\zeta(r_i)) = \Upsilon(\kappa(r_i))$, if $\frac{k}{2j_R - j_{i_0}} = \tau(\Sigma(\zeta(r_i)), \{\frac{0}{2j_R - j_{i_0}}, \ldots, \frac{2j_R - j_{i_0}}{2j_R - j_{i_0}}\})$, then $\frac{k}{2j_R - j_{i_0}} = \tau(\Upsilon(\kappa(r_i)), \{\frac{0}{2j_R - j_{i_0}}, \ldots, \frac{2j_R - j_{i_0}}{2j_R - j_{i_0}}\})$. Thus $\gamma(i) = \frac{k}{2j_R - j_{i_0}} = \lambda(i)$. If $k \leq j_R$, $F(i) = \beta(k)$ and $G(i) = \alpha(k)$. By the choice of $(\alpha(k), \beta(k))$, $\mu((\alpha(k), \beta(k))) \in \mathrm{cl}_X(U_0)$. Therefore, $h(i) \in \mathrm{cl}_X(U_0)$. If $j_R < k$, $F(i) = \beta(2j_R - k)$ and $G(i) = \alpha(2j_R - k)$. Thus $h(i) = \mu(\alpha(2j_R - k), \beta(2j_R - k)) \in \mathrm{cl}_X(U_0)$. Therefore, if $i \in \{1, \ldots, N\}$, $h(i) \in \mathrm{cl}_X(U_0)$.

Case 2. $i \in \{N+1, \ldots, 2N\}$.

Let $\frac{k}{j_R - j_{i_0}} = \tau(\Lambda(\Pi(\zeta(r_{2N-i+1}))), \{\frac{0}{j_R - j_{i_0}}, \ldots, \frac{j_R - j_{i_0}}{j_R - j_{i_0}}\}) = \gamma(i)$, $\frac{l}{2j_R - j_{i_0}} = \tau(\Upsilon(\kappa(r_{2N-i+1})), \{\frac{0}{2j_R - j_{i_0}}, \ldots, \frac{2j_R - j_{i_0}}{2j_R - j_{i_0}}\}) = \lambda(2N - i + 1)$. So, $F(i) = \beta(j_R - k)$.

If $l \leq j_R$, then $G(i) = \alpha(l)$. Thus $h(i) = \mu(\beta(j_R - k), \alpha(l))$. Since $\Sigma \circ \zeta = \Upsilon \circ \kappa$, we can apply Claim 17 and obtain that $d(\beta(j_R - k), \beta(l)) < 3\delta$. Then $D((\beta(j_R - k), \alpha(l)), (\beta(l), \alpha(l))) < \frac{3}{2}\delta$. By the choice of δ, $d(\mu(\beta(j_R - k), \alpha(l)), \mu(\beta(l), \alpha(l))) < \frac{1}{16}(\text{diameter}(Y))$. By the choice of $(\beta(l), \alpha(l))$, we obtain $\mu(\beta(l), \alpha(l)) \in \mathrm{cl}_X(U_0)$. Therefore, $d(h(i), \mathrm{cl}_X(U_0)) < \frac{1}{16}(\text{diameter}(Y))$.

If $j_R < l$, then $G(i) = \alpha(2j_R - l)$. Thus $h(i) = \mu(\beta(j_R - k), \alpha(2j_R - l))$. Applying Claim 17, we obtain $d(\beta(j_R - k), \beta(2j_R - l)) < 3\delta$. By the choice of δ and $(\beta(2j_R - l), \alpha(2j_R - l))$, $d(h(i), \mathrm{cl}_X(U_0)) < \frac{1}{16}(\text{diameter}(Y))$.

Case 3. $i \in \{2N+1, \ldots, 3N\}$.

Let $\frac{l}{j_R - j_{i_0}} = \tau(\Lambda(F(\kappa(r_{i-2N}))), \{\frac{0}{j_R - j_{i_0}}, \ldots, \frac{j_R - j_{i_0}}{j_R - j_{i_0}}\}) = \lambda(i)$ and $\frac{k}{2j_R - j_{i_0}} = \tau(\Sigma(\zeta(r_{i-2N})), \{\frac{0}{2j_R - j_{i_0}}, \ldots, \frac{2j_R - j_{i_0}}{2j_R - j_{i_0}}\}) = \gamma(i - 2N)$. Then $G(i) = \beta(j_R - l)$.

If $k \leq j_R$, then $F(i) = \beta(k)$. Thus $h(i) = \mu(\beta(k), \beta(j_R - l))$. Since $\Sigma \circ \zeta = \Upsilon \circ \kappa$, we can apply Claim 18 and obtain that $d(\beta(j_R - l), \alpha(k)) < 3\delta$. Then $D((\beta(j_R - l), \beta(k)), (\alpha(k), \beta(k))) < \frac{3}{2}\delta$. By the choice of δ, $d(\mu(\beta(j_R - l), \beta(k)), \mu(\alpha(k), \beta(k))) < \frac{1}{16}(\text{diameter}(Y))$. By the choice of $(\alpha(k), \beta(k))$, we have $\mu(\alpha(k), \beta(k)) \in \mathrm{cl}_X(U_0)$. Therefore, $d(h(i), \mathrm{cl}_X(U_0)) < \frac{1}{16}(\text{diameter}(Y))$.

If $j_R < k$, then $F(i) = \beta(2j_R - k)$. Thus $h(i) = \mu(\beta(2j_R - k), \beta(j_R - l))$. Applying Claim 18, we obtain $d(\beta(j_R - l), \alpha(2j_R - k)) < 3\delta$. By the choice of δ and $(\alpha(2j_R - k), \beta(2j_R - k))$, $d(h(i), \mathrm{cl}_X(U_0)) < \frac{1}{16}(\text{diameter}(Y))$.

Case 4. $i \in \{3N + 1, \ldots, 4N\}$.

Let $\frac{k}{j_R - j_{i_0}} = \tau(\Lambda(\Pi(\zeta(r_{4N-i+1}))), \{\frac{0}{j_R - j_{i_0}}, \frac{1}{j_R - j_{i_0}}, \ldots, \frac{j_R - j_{i_0}}{j_R - j_{i_0}}\}) = \gamma(i - 2N)$ and $\frac{k'}{j_R - j_{i_0}} = \tau(\Lambda(\digamma(\kappa(r_{4N-i+1}))), \{\frac{0}{j_R - j_{i_0}}, \frac{1}{j_R - j_{i_0}}, \ldots, \frac{j_R - j_{i_0}}{j_R - j_{i_0}}\}) = \lambda(6N - i + 1)$. Then $F(i) = \beta(j_R - k)$ and $G(i) = \beta(j_R - k')$.

Let l be such that $\frac{l}{2j_R - j_{i_0}} = \tau(\Sigma(\zeta(r_{4N-i+1})), \{\frac{0}{2j_R - j_{i_0}}, \ldots, \frac{2j_R - j_{i_0}}{2j_R - j_{i_0}}\}) = \tau(\Upsilon(\kappa(r_{4N-i+1})), \{\frac{0}{2j_R - j_{i_0}}, \ldots, \frac{2j_R - j_{i_0}}{2j_R - j_{i_0}}\})$. If $l \le j_R$, by Claim 17, $d(\beta(j_R - k), \beta(l)) < 3\delta$, and by Claim 18, $d(\beta(j_R - k'), \alpha(l)) < 3\delta$. Thus $D((\beta(j_R - k), \beta(j_R - k')), (\beta(l), \alpha(l))) < 3\delta$. By the choice of δ, $d(\mu(\beta(j_R - k), \beta(j_R - k')), \mu(\beta(l), \alpha(l))) < \frac{1}{16}(\text{diameter}(Y))$. Hence, $d(h(i), \mathrm{cl}_X(U_0)) < \frac{1}{16}(\text{diameter}(Y))$. If $j_R < l$, by Claim 17, $d(\beta(j_R - k), \beta(2j_R - l)) < 3\delta$, and by Claim 18, $d(\beta(j_R - k'), \alpha(2j_R - l)) < 3\delta$. This implies that $d(h(i), \mathrm{cl}_X(U_0)) < \frac{1}{16}(\text{diameter}(Y))$.

We have proved Claim 19.

Claim 20. $\text{diameter}(h(L)) < \frac{1}{2}(\text{diameter}(Y))$.

Claim 20 follows from the fact that $\text{diameter}(\mathrm{cl}_X(U_0)) < \delta_1 < \frac{1}{16}(\text{diameter}(Y))$ and Claim 19.

Since Claims 16 and 20 are contradictory, we have finished the proof of Theorem 1. ■

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References

[1] P. Bacon, *An acyclic continuum that admits no mean*, Fund. Math. 67 (1970), 11-13.

[2] P. Bacon, *Unicoherence in means*, Colloq. Math. 21 (1970), 211-215.

[3] M. Bell and A. Watson, *Not all dendroids have means*, Houston J. Math. 22 (1996), 39-50.

[4] D. Bellamy, *Nonexistence of means on Knaster continua*, preprint.

[5] F. Capulín and W. J. Charatonik, *Retractions from $C(X)$ onto X and continua of type N*, preprint.

[6] J. J. Charatonik, W. J. Charatonik, K. Omiljanowski and J. R. Prajs, *Hyperspace retractions for curves*, Dissertationes Math. (Rozprawy Mat.) 370 (1990), 1-34.

[7] A. Illanes, *Means on compactifications of the ray*, preprint.

[8] A. Illanes, *The buckethandle continuum admits no mean*, Continuum Theory (Lecture Notes in Pure and Applied Mathematics Series 230), Marcel Dekker, New York and Basel, 2002, 137-142.

[9] A. Illanes and S. B. Nadler, Jr., *Hyperspaces, Fundamentals and Recent Advances*, Pure and Applied Mathematics 216, Marcel Dekker, Inc., New York, (1999).

[10] A. Illanes and L. C. Simón, *Means with special properties*, Houston J. Math. 29 (2003), 313–324.

[11] K. Kawamura and E. D. Tymchatyn, *Continua which admit no mean*, Colloq. Math. 71 (1996), 97-105.

[12] K. Kuratowski, *Topology*, vol. 2, Academic Press and PWN, New York, London and Warzawa, 1968.

[13] S. B. Nadler, Jr., *Continuum Theory, An Introduction*, Pure and Applied Mathematics 158, Marcel Dekker, Inc., New York, Basel, Hong Kong, (1992).

[14] R. Sikorski and K. Zarankiewicz, *On uniformization of functions (I)*, Fund. Math. 41 (1954), 339-344.

[15] A. D. Wallace, *Acyclicity of compact connected semigroups*, Fund. Math. 50 (1961), 99-105.

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