AN UNCOUNTABLE COLLECTION OF DENDROIDS MUTUALLY INCOMPARABLE BY CONTINUOUS FUNCTIONS

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ABSTRACT. We answer a question of B. Knaster by constructing an uncountable collection of dendroids whose members are not comparable by continuous maps.

1. INTRODUCTION

Every space considered in this paper is metric and every map is continuous. If x is a point of a space X and r is a positive number, by $B(x,r)$ we denote the open ball in X with center x and radius r. By a *continuum* we understand a connected compact space. A space X is *arcwise connected* if for every $a,b \in X$ there is an arc $A \subset X$ such that $a,b \in A$. (An arc is a continuum homeomorphic to the unit interval $[0,1] \subset \mathbb{R}$.) A continuum X is *unicoherent* if $A \cap B$ is connected for each subcontinua A and B such that $A \cup B = X$. X is *hereditarily unicoherent* if every subcontinuum of X is unicoherent. A *dendroid* is an arcwise connected hereditarily unicoherent continuum. A *fan* is dendroids with only one ramification point. By a result of H. Cook [5], dendroids can be characterized as arcwise connected tree-like continua (inverse limits of trees). For every two points a and b in a dendroid X, there is exactly one arc in X containing a and b as its endpoints. We will denote this arc by $\langle a,b\rangle$. If it is convenient, we will assume that $\langle a,b\rangle$ is ordered from a to b. Recall that every subcontinuum of a dendroid is a dendroid. For more information on dendroids see [3] and [9].

We say that two continua are *comparable* by continuous maps if one of those continua can be mapped onto the other. Otherwise, the continua are *incomparable*. In 1932, Z. Waraszkiewicz [13] constructed a collection of $2^{\aleph_0}$ incomparable plane continua. In 1967, H. Cook [4] proved that there are $2^{\aleph_0}$ incomparable solenoids. In 1971, D. P. Bellamy constructed $2^{\aleph_0}$ incomparable chainable continua. (A continuum is chainable if it is homeomorphic to an inverse limit of arcs.) Other interesting examples of uncountable collections of incomparable continua were given by R. L. Russo [12], W. T. Ingram [6], T. Maćkowiak [8] and M. M. Awartani [1]. None of the of the collections mentioned above contains arcwise connected continua. For instance, each continuum in the original Waraszkiewicz collection is a spiral converging to the unit circle, first by a certain number of clockwise rotations, then by a number of counterclockwise rotations, then again by another number of clockwise rotations, and so on. The Waraszkiewicz spirals are not only incomparable with each other. In 1999, J. Prajs and A. Swół [11] observed that the Waraszkiewicz spirals are incomparable with most continua in the following sense. Prajs and Swół

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added to the Waraszkiewicz collection a single continuum L and proved that for any continuum X on of the following is true: (a) X is incomparable with some Waraszkiewicz spiral, (b) X incomparable with L, (c) X is comparable with all continua.

In 1961, B. Knaster [7] asked if an uncountable collection of incomparable dendroids could be found. The same question is also listed in the electronic problem collection in continuum theory by J. Prajs [10, Problem 34]. In this paper, we answer the question of Knaster by constructing $2^{\aleph_0}$ incomparable dendroids. In our construction, we follow R. L. Russo who replaced the unit circle in the Waraszkiewicz spirals by a simple triod. (A simple triod is a space homeomorphic to the letter Y.) We cannot use spirals, since dendroids are arcwise connected. We go around this problem in the following way. All dendroids in our collection are retracts of the same dendroid D which is the union of a simple triod Y and a sequence of arcs $L_1, L_2, \ldots$ Each of the arcs intersects Y only in one point p which is the center of Y. Any two of the arcs intersect only at p. The arc L_n starts at p and goes 2^n times clockwise around Y, then continues 2^n times counterclockwise, and then goes 2^n times clockwise again. We denote the other end of L_n by p_n. The arcs converge to Y as n goes to infinity. We construct and uncountable collection $\mathcal{D}$ of retracts of D by taking the union of Y and only some of the arcs. We do it in such a way that for any two $D_0, D_1 \in \mathcal{D}$ there is infinitely many of the arcs included in D_1 but not in D_0. The idea of our construction is based on the following observation. We prove (see Theorem 3.1) that if f is a map from a continuum $X \subset D$ into D and n is sufficiently large, then $f^{-1}(p_n) \subset L_n$. In other words, $p_n \notin f(D \setminus L_n)$. Now, if $X \subset D_0$ and L_n is one of the arcs included in D_1 but not in D_0, $p_n \notin f(X)$. This proves that the elements of $\mathcal{D}$ not only incomparable but none of them is an image of a a subcontinuum of any other. Observe that D and each member of $\mathcal{D}$ is a fan since p is the only ramification point of D.

2. Dendroid D

In this section we will define the dendroid D as it was outlined in the introduction. Our construction will be made in $\mathbb{R}^3$. For any $x, y \in \mathbb{R}$, the straight linear segment in $\mathbb{R}^3$ will be denoted by $[x, y]$.

Let $p = (0,0,0)$, $a_0 = (1,0,0)$, $a_1 = (\cos 2\pi/3, \sin 2\pi/3, 0)$, $a_2 = (\cos 4\pi/3, \sin 4\pi/3, 0)$ and $Y = [a_0, p] \cup [a_1, p] \cup [a_2, p]$. For each $j = 0, 1, \ldots$, set $z_j = 2 - 2^{-j}$. Observe that (z_j) is a strictly increasing sequence with values between 1 and 2. For each $n = 1, 2, \ldots$ and each $j = 0, \ldots, 6 \cdot 2^{n+2}$ we will define a point $s_{n,j} \in \mathbb{R}^3$ in the following way. Set $s_{n,j} = (0, 0, 2^{-n} z_j)$ if j is odd. For each $k = 0, \ldots, 2^{n+2}$, set $s_{n,6k} = (1, 0, 2^{-n} z_{6k})$,

$$s_{n,6k+2} = \begin{cases} (\cos 2\pi/3, \sin 2\pi/3, 2^{-n} z_{6k+2}), & \text{if } k = 0, \ldots, 2^n - 1 \text{ or} \\ & k = 2^{n+1}, \ldots, 2^{n+2} - 1; \\ (\cos 4\pi/3, \sin 4\pi/3, 2^{-n} z_{6k+2}), & \text{if } k = 2^n, \ldots, 2^{n+1} - 1; \end{cases}$$

and

$$s_{n,6k+4} = \begin{cases} (\cos 4\pi/3, \sin 4\pi/3, 2^{-n} z_{6k+4}), & \text{if } k = 0, \ldots, 2^n - 1 \text{ or} \\ & k = 2^{n+1}, \ldots, 2^{n+2} - 1; \\ (\cos 2\pi/3, \sin 2\pi/3, 2^{-n} z_{6k+4}), & \text{if } k = 2^n, \ldots, 2^{n+2} - 1. \end{cases}$$

Set

$$L_n = [p, s_{n,0}] \cup \bigcup_{j=1}^{2^{n+2}} [s_{n,j-1}, s_{n,j}].$$

The following two propositions readily follow from the construction.

Proposition 2.1. *For each $n = 1, 2, \dots$*

(1) *L_n is an arc with endpoints p and $p_n = s_{n,2^{n+2},0}$,*
(2) *$Y \cap L_n = \{p\}$, and*
(3) *$L_m \cap L_n = \{p\}$ for each positive integer $m \neq n$.*

□

Let $D = Y \cup \bigcup_{n=1}^{\infty} L_n$.

Proposition 2.2. *D is a dendroid.* □

Let r be the projection of $\mathbb{R}^3$ onto the xy-plane restricted to D. Observe that r is a retraction of D onto Y.

The retraction r restricted to each of the segments $[p, s_{n,0}]$, $[s_{n,j-1}, s_{n,j}]$ ($j = 1, \dots, 6 \cdot 2^{n+2}$) is a homeomorphism onto one of the segments $[p, a_0]$, $[p, a_1]$ and $[p, a_2]$. Observe that each of the last three segments has length 1. Let $\sigma_n : \left[-1, 6 \cdot 2^{n+2}\right] \to L_n$ be the parametrization of L_n such that

- $\sigma_n(-1) = p$,
- $\sigma_n\left(6 \cdot 2^{n+2}\right) = p_n$, and
- $r \circ \sigma_n$ restricted to the interval $[j-1, j]$ (where $j = 0, \dots, 6 \cdot 2^{n+2}$) is a length preserving homeomorphism onto one of the segments $[p, a_0]$, $[p, a_1]$ and $[p, a_2]$.

The following three propositions are again easy consequence of the construction.

Proposition 2.3. (1) *$r(\sigma_n(6k)) = a_0$ for $k = 0, \dots 2^{n+2}$,*
(2) *$r(\sigma_n(6k+2)) = a_1$ for $k = 0, \dots 2^n - 1$ and $k = 2^{n+1}, \dots 2^{n+2} - 1$,*
(3) *$r(\sigma_n(6k+4)) = a_1$ for $k = 2^n, \dots 2^{n+1} - 1$,*
(4) *$r(\sigma_n(6k+4)) = a_2$ for $k = 0, \dots 2^n - 1$ and $k = 2^{n+1}, \dots 2^{n+2} - 1$,*
(5) *$r(\sigma_n(6k+2)) = a_2$ for $k = 2^n, \dots 2^{n+1} - 1$.* □

Proposition 2.4. *$r(\sigma_n(q)) \neq r(\sigma_n(q-2))$ for each even integer $q = 2, \dots, 6 \cdot 2^{n+2}$.* □

For $m = 0, 1, 2$ let m^+ denote $m + 1 \mod 3$, and let m^- denote $m - 1 \mod 3$. So, $0^+ = 1$, $1^+ = 2$, $2^+ = 0$, $2^- = 1$, $1^- = 0$ and $0^- = 2$.

Proposition 2.5. *Suppose $r(\sigma_n(q)) = a_m$. Then q is an even integer between 0 and $6 \cdot 2^{n+2}$, and*

(1) *$r(\sigma_n(q+2)) = a_{m^+}$ for $q = 0, \dots, 6 \cdot 2^n - 2, 6 \cdot 2^{n+1}, \dots, 6 \cdot 2^{n+2} - 2$,*
(2) *$r(\sigma_n(q+2)) = a_{m^-}$ for $q = 6 \cdot 2^n, \dots, 6 \cdot 2^{n+1} - 2$,*
(3) *$r(\sigma_n(q-2)) = a_{m^-}$ for $q = 2, \dots, 6 \cdot 2^n, 6 \cdot 2^{n+1} + 2, \dots, 6 \cdot 2^{n+2}$, and*
(4) *$r(\sigma_n(q-2)) = a_{m^+}$ for $q = 6 \cdot 2^n + 2, \dots, 6 \cdot 2^{n+1}$.* □

3. Maps from a subcontinuum of D into D

In this section we will prove the following theorem.

Theorem 3.1. *Suppose X is a subcontinuum of D and f is a map of X into D. Then, there is a positive integer ν such that*

$$f^{-1}(p_n) \subset L_n \setminus \{p\}$$

for each $n \geq \nu$.

Proof. We may assume that $p \in X$, because otherwise X would be locally connected and $f(X)$ would contain only finitely of the endpoints p_n, making the theorem trivially true. Since D is a dendroid, X is also a dendroid. Hence, $\langle x, p\rangle \subset X$ for every $x \in X$.

We will use the following notation:

$F = f^{-1}(a_0) \cup f^{-1}(a_1) \cup f^{-1}(a_2)$,
$F_0 = f^{-1}(a_1) \cup f^{-1}(a_2)$,
$F_1 = f^{-1}(a_0) \cup f^{-1}(a_2)$, and
$F_2 = f^{-1}(a_0) \cup f^{-1}(a_1)$.

For each $\tau = 0, 1, 2$, we will define a finite set $T_\tau \subset F$ in the following way:

Step 0: If $[p, a_\tau] \cap F = \emptyset$, we conclude the construction by setting $T_\tau = \emptyset$. Otherwise, if $[p, a_\tau] \cap F \neq \emptyset$, we proceed to the next step.

Step 1: Let $t_1^{(\tau)}$ be the first point in the segment $[p, a_\tau]$ belonging to F. Let $\mu(\tau, 1) \in \{0, 1, 2\}$ be such that $f\left(t_1^{(\tau)}\right) = a_{\mu(\tau,1)}$. If $\left[t_1^{(\tau)}, a_\tau\right] \cap F_{\mu(\tau,1)} = \emptyset$, we conclude the construction by setting $T_\tau = \left\{t_1^{(\tau)}\right\}$. Otherwise, we proceed to the next step.

Step 2: Let $t_2^{(\tau)}$ be the first point in the segment $\left[t_1^{(\tau)}, a_\tau\right]$ belonging to $F_{\mu(\tau,1)}$. Let $\mu(\tau, 2) \in \{0, 1, 2\}$ be such that $f\left(t_2^{(\tau)}\right) = a_{\mu(\tau,2)}$. If $\left[t_2^{(\tau)}, a_\tau\right] \cap F_{\mu(\tau,2)} = \emptyset$, we conclude the construction by setting $T_\tau = \left\{t_1^{(\tau)}, t_2^{(\tau)}\right\}$. Otherwise, we proceed to the next step.

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Step k: Let $t_k^{(\tau)}$ be the first point in the segment $\left[t_{k-1}^{(\tau)}, a_\tau\right]$ belonging to $F_{\mu(\tau,k-1)}$. Let $\mu(\tau, k) \in \{0, 1, 2\}$ be such that $f\left(t_k^{(\tau)}\right) = a_{\mu(\tau,k)}$. If $\left[t_k^{(\tau)}, a_\tau\right] \cap F_{\mu(\tau,k)} = \emptyset$, we end the construction by setting $T_\tau = \left\{t_1^{(\tau)}, t_2^{(\tau)}, \dots, t_k^{(\tau)}\right\}$. Otherwise, we proceed to the next step.

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Since X is compact and f is continuous, the construction of T_τ must end in some step. We will denote the number of elements of T_τ by $k(\tau)$.

The next proposition summarizes the properties of the construction.

Proposition 3.2. *For each $\tau = 0, 1, 2$, the following statements are true:*

(1) $[p, a_\tau] \cap F = \emptyset$ *if* $T_\tau = \emptyset$.
(2) *For each* $j = 1, \dots, k(\tau)$*:*
   (a) $t_j^{(\tau)} \in [p, a_\tau] \cap X$,
   (b) $f\left(t_j^{(\tau)}\right) = a_{\mu(\tau,j)}$ *where* $\mu(\tau, j) = 0, 1, 2$.
(3) *For each* $j = 1, \dots, k(\tau) - 1$*:*
   (a) $t_{j+1}^{(\tau)} \in \left[t_j^{(\tau)}, a_\tau\right]$,

(b) $f\left(\left[t_j^{(\tau)}, t_{j+1}^{(\tau)}\right) \cap F\right) = \left\{a_{\mu(\tau,j)}\right\}$,
(c) $\mu(\tau, j) \neq \mu(\tau, j+1)$.

(4) $\left[t_{k(\tau)}^{(\tau)}, a_\tau\right] \cap F_{\mu(\tau,k(\tau))} = \emptyset$ *if* $k(\tau) > 0$. □

Let K be the maximum of $k(0)$, $k(1)$ and $k(2)$. Set $T = T_0 \cup T_1 \cup T_2$. If $t = t_j^{(\tau)} \in T$ then we will denote $\mu(\tau, j)$ by $\mu[t]$. (Observe that if $p \in T$ then $p = t_1^{(0)} = t_1^{(1)} = t_1^{(2)}$ and $\mu(0,1) = \mu(1,1) = \mu(2,1)$.)

For each $j = 1, \ldots, k(\tau)-1$, let $v_j^{(\tau)}$ be the last point in $\left[t_j^{(\tau)}, t_{j+1}^{(\tau)}\right)$ that belongs to $f^{-1}\left(a_{\mu(\tau,j)}\right)$. Set $v_{k(\tau)}^{(\tau)} = a_\tau$. Let $\mathcal{V} = \{V(t) : t \in T\}$ be a collection of mutually disjoint connected open subsets of Y such that

- $\left[t_j^{(\tau)}, v_j^{(\tau)}\right] \subset V\left(t_j^{(\tau)}\right)$ and
- $p \in V(t)$ if and only if $p = t$.

For $m = 0, 1, 2$, let V_m be the union of all sets $V(t)$ with $\mu[t] = m$. Observe that $f^{-1}(a_m) \cap Y \subset V_m$ and V_0, V_1 and V_2 are mutually disjoint. Set $A_m = Y \setminus V_m$. Clearly, $A_0 \cup A_1 = A_0 \cup A_2 = A_1 \cup A_2 = Y$. Since A_m is compact, there is a positive number $\eta < 0.5$ such that $f(A_m \cap X) \cap B(a_m, 2\eta) = \emptyset$ for $m \in \{0, 1, 2\}$. There exists a positive integer i_1 such that

$$f(x) \notin B(a_m, \eta)$$

for each $m \in \{0, 1, 2\}$, each $n \geq i_1$ and each $x \in L_n \cap X$ such that $r(x) \in A_m$.

Since X is compact, there is a positive number δ such that $|f(x) - f(x')| < \eta$ for all $x, x' \in X$ such that $|x - x'| < \delta$. Let $\kappa \geq i_1$ be an integer such that $|z - r(z)| < \delta$ for each $z \in Y \cup \bigcup_{i=\kappa}^{\infty} L_i$.

Denote by X_0 the intersection of $Y \cup \bigcup_{i=1}^{\kappa-1} L_i$ with X. As X_0 is locally connected, $f(X_0)$ is also locally connected and, therefore, it may contain only finitely many endpoints p_n's. Let n_1 be an integer such that $p_n \notin f(X_0)$ for each $n \geq n_1$.

Since r is a retraction of D onto Y, there is a positive integer $\nu \geq n_1$ such that $|z - r(z)| < \eta$ for each $z \in Y \cup \bigcup_{n=\nu}^{\infty} L_n$. By increasing ν if necessary, we may assume that $\nu > 4$, $2^{\nu-1} > K$ and $f(p) \notin L_n \setminus \{p\}$ for each $n \geq \nu$.

Suppose that there is an integer $n \geq \nu$ such that $p_n \in f(X)$. Let $u \in X$ be an arbitrary point such $f(u) = p_n$. By the choice of n_1, $u \in L_i$ for $i \geq \kappa$. To complete Theorem 3.1 we must show that $i = n$.

Let w be the first point in the arc $\langle u, p\rangle$ (oriented from u to p) such that $f(w) = p$. Set $L = \langle u, w\rangle$. Clearly, $L \subset X$ and $f(L) = L_n$.

Proposition 3.3. *Suppose* $x \in \langle u, p\rangle$ *and* $r(x) \in A_m$ *for some* $m = 0, 1, 2$. *Then,* $f(x) \neq \sigma_n(q)$ *for any* $q = 0, \ldots 6 \cdot 2^{n+2}$ *such that* $r(\sigma_n(q)) = a_m$.

Proof. Suppose to the contrary that $f(x) = \sigma_n(q)$ and $r(\sigma_n(q)) = a_m$ for some $q = 0, \ldots 6 \cdot 2^{n+2}$. Thus, $r(f(x)) = a_m$ and $f(x) \in B(a_m, \eta)$ by the choice of ν. But, $f(x) \notin B(a_m, \eta)$ by the choice of i_1. □

Proposition 3.4. *Suppose that*

- $q = 2, \ldots, 6 \cdot 2^{n+2} - 2$ *is an even integer such that* $r(\sigma_n(q)) = a_m$, $r(\sigma_n(q+2)) = a_{m'}$ *and* $r(\sigma_n(q-2)) = a_{m''}$ *for some* $m, m', m'' = 0, 1, 2$,
- $y', y'' \in L$ *are such that* $f(y') = \sigma_n(q+2)$ *and* $f(y'') = \sigma_n(q-2)$.

Then, there is $y \in \langle y', y''\rangle$ *such that* $r(y) \in T$ *and* $f(r(y)) = a_m$.

Proof. There is a point $y_0 \in \langle y', y'' \rangle$ such that $f(y_0) = \sigma_n(q)$. By 3.3, $r(y_0) \in V_m$, $r(y') \in V_{m'}$ and $r(y'') \in V_{m''}$. By 2.4, $m \neq m'$ and $m \neq m''$. Therefore, there is $t \in T$ such that $r(y_0) \in V(t)$, $r(y') \notin V(t)$ and $r(y'') \notin V(t)$. Let C be the closure of a component of $\langle y', y'' \rangle \setminus r^{-1}(\{p, a_0, a_1, a_2\})$ such that $y_0 \in C$. $r(C)$ is a straight linear segment whose endpoints are in the set $\{p, a_0, a_1, a_2, r(y'), r(y'')\}$. It follows that either $t = p$ and $p \in r(C)$ or $V(t) \subset r(C)$. The proposition is true in either of the cases. □

Corollary 3.5. *$r(L) \cap T \neq \emptyset$ and, therefore, $K > 0$.* □

Proposition 3.6. *Suppose $x \in L$ such that $f(r(x)) = a_m$ for some $m = 0, 1, 2$. Let $v = {\sigma_n}^{-1}(f(x))$. Then, there is an even integer $q(x)$ such that $|q(x) - v| < \eta$. Moreover,*

(1) *If $f(r(x)) = a_0$, then $q(x) = 6j$ for some $j = 0, \ldots, 2^{n+2}$.*
(2) *If $f(r(x)) = a_1$, then either*
  - *$q(x) = 6j + 2$ for some $j = 0, \ldots, 2^n - 1, 2^{n+1}, \ldots, 2^{n+2} - 1$, or*
  - *$q(x) = 6j + 4$ for some $j = 2^n, \ldots, 2^{n+1} - 1$.*
(3) *If $f(r(x)) = a_2$, then either*
  - *$q = 6j + 4$ for some $j = 0, \ldots, 2^n - 1, 2^{n+1}, \ldots, 2^{n+2} - 1$, or*
  - *$q = 6j + 2$ for some $j = 2^n, \ldots, 2^{n+1} - 1$.*

Proof. Suppose that $f(r(x)) = a_0$. It follows from the choice of κ that

$$|f(x) - f(r(x))| = |f(x) - a_0| < \eta.$$

Since r does not increase the distance,

$$|r(f(x)) - a_0| = |r(f(x)) - r(a_0)| \leq |f(x) - a_0| < \eta.$$

Since $\eta < 0.5$, $r(\sigma_n(v)) = r(f(x)) \in (p, a_0]$. Let k be an integer such that $v \in [k-1, k] \subset \left[-1, 6 \cdot 2^{n+2}\right]$. By the definition of σ_n, $r \circ \sigma_n$ restricted to the interval $[k-1, k]$ is a length preserving homeomorphism onto $[p, a_0]$. It follows from 2.3 that either $k - 1$ or k must be equal $6j$ for some $j = 0, \ldots 2^{n+2}$. Since $r \circ \sigma_n$ restricted to the interval $[k-1, k]$ is preserves length, $|6j - v| < \eta$. This completes the proof of the proposition in the case $f(r(x)) = a_0$. The proofs for $f(r(x)) = a_1$ and $f(r(x)) = a_2$ are essentially the same and will be omitted. □

We will use $q(x)$ to denote the integer defined in 3.6 for each x satisfying the hypothesis of 3.6. Notice that $q(x)$ is unique. Observe also that if $v \in [k_1, k_2]$ for some integers k_1, k_2, then $q(x) \in [k_1, k_2]$.

Lemma 3.7. *Suppose that*

- *m, m' and m'' are the numbers 0, 1 and 2 (not necessarily in the same order),*
- *$x_0 \in L$, $x_1 \in \langle x_0, p \rangle$,*
- *$f(r(x_0)) = f(r(x_1)) = a_m$,*
- *$r(\langle x_0, x_1 \rangle) \subset A_{m'} \cap A_{m''}$, and*
- *$q(x_0) > 0$.*

Then, $x_1 \in L$ and $q(x_1) = q(x_0)$.

Proof. By 2.4, $r(\sigma_n(q(x_0) - 2)) \neq r(\sigma_n(q(x_0))) = a_m$. Thus, $r(\sigma_n(q(x_0) - 2))$ is either to $a_{m'}$ or $a_{m''}$. Since $r(\langle x_0, x_1 \rangle) \subset A_{m'} \cap A_{m''}$, Proposition 3.3 implies that

$$\sigma_n(q(x_0) - 2) \notin f(\langle x_0, x_1 \rangle).$$

Similarly, one may prove that

$$\sigma_n\left(q\left(x_0\right)+2\right) \notin f\left(\left\langle x_0, x_1\right\rangle\right)$$

in the case $q\left(x_0\right) < 6 \cdot 2^{n+2}$. As $f\left(\left\langle x_0, x_1\right\rangle\right)$ is connected, it must be contained in $\left[\sigma_n\left(q\left(x_0\right)-2\right), \sigma_n\left(q\left(x_0\right)+2\right)\right]$ (or in $\left[\sigma_n\left(q\left(x_0\right)-2\right), \sigma_n\left(q\left(x_0\right)\right)\right]$ in the case $q\left(x_0\right) = 6 \cdot 2^{n+2}$). It follows that $x_1 \in L$ and $q\left(x_1\right) \in \left[q\left(x_0\right)-2, q\left(x_0\right)+2\right]$. Since $q\left(x_0\right)$ is the only point in the interval $\left[q\left(x_0\right)-2, q\left(x_0\right)+2\right]$ which is mapped by $r \circ \sigma_n$ to a_m. Thus, $q\left(x_1\right) = q\left(x_0\right)$. □

Let $g : \left[0, 6 \cdot 2^{n+2}\right] \to \left[0, 6 \cdot 2^n\right]$ be defined by

$$g(z) = \begin{cases} z, & \text{if } 0 \leq z \leq 6 \cdot 2^n, \\ 6 \cdot 2^{n+1} - z, & \text{if } 6 \cdot 2^n \leq z \leq 6 \cdot 2^{n+1}, \\ z - 6 \cdot 2^{n+1}, & \text{if } 6 \cdot 2^{n+1} \leq z \leq 6 \cdot 2^{n+2}. \end{cases}$$

Lemma 3.8. *Suppose that*

- *m, m' and m'' are the numbers 0, 1 and 2 (not necessarily in the same order),*
- *$x_0 \in L$, $x_1 \in \left\langle x_0, p\right\rangle$,*
- *$f\left(r\left(x_0\right)\right) = a_m$,*
- *$f\left(r\left(x_1\right)\right) = a_{m'}$,*
- *$r\left(\left\langle x_0, x_1\right\rangle\right) \subset A_{m''}$, and*
- *$q\left(x_0\right) > 2$ (or $q\left(x_0\right) = 2$, $m = 1$ and $m' = 2$).*

Then, $x_1 \in L$ and $q\left(x_1\right)$ is either $q\left(x_0\right)-2$ or $q\left(x_0\right)+2$. Moreover,

1. *$g\left(q\left(x_1\right)\right) = g\left(q\left(x_0\right)\right)+2$ if $m' = m^+$, and*
2. *$g\left(q\left(x_1\right)\right) = g\left(q\left(x_0\right)\right)-2$ if $m' = m^-$.*

Proof. The proofs of (1) and (2) are similar. We will leave the proof of (2) to the reader. We will consider the following cases separately:

(i) $q\left(x_0\right) = 2, \ldots, 6 \cdot 2^n - 4, 6 \cdot 2^{n+1} + 2, \ldots, 6 \cdot 2^{n+2} - 4$,
(ii) $q\left(x_0\right) = 6 \cdot 2^n - 2$,
(iii) $q\left(x_0\right) = 6 \cdot 2^{n+2} - 2$,
(iv) $q\left(x_0\right) = 6 \cdot 2^n$ or $q\left(x_0\right) = 6 \cdot 2^{n+2}$,
(v) $q\left(x_0\right) = 6 \cdot 2^n + 4, \ldots, 6 \cdot 2^{n+1} - 2$,
(vi) $q\left(x_0\right) = 6 \cdot 2^n + 2$, and
(vii) $q\left(x_0\right) = 6 \cdot 2^{n+1}$.

We will show that (iv) cannot occur. In each of the remaining cases we will specify the value(s) for $q\left(x_1\right)$. It will be obvious that $g\left(q\left(x_1\right)\right) = g\left(q\left(x_0\right)\right)+2$.
Case (i): By Proposition 2.5, $r\left(\sigma_n\left(q\left(x_0\right)-2\right)\right) = a_{m''}$, $r\left(\sigma_n\left(q\left(x_0\right)+2\right)\right) = a_{m'}$ and $r\left(\sigma_n\left(q\left(x_0\right)+4\right)\right) = a_{m''}$. Since $r\left(\left\langle x_0, x_1\right\rangle\right) \subset A_{m''}$, Proposition 3.3 implies that neither of the points $\sigma_n\left(q\left(x_0\right)-2\right)$ and $\sigma_n\left(q\left(x_0\right)+4\right)$ belongs to $f\left(\left\langle x_0, x_1\right\rangle\right)$. As $f\left(\left\langle x_0, x_1\right\rangle\right)$ is connected, it must be contained in $\left[\sigma_n\left(q\left(x_0\right)-2\right), \sigma_n\left(q\left(x_0\right)+4\right)\right]$. It follows that $x_1 \in L$ and $q\left(x_1\right) \in \left[q\left(x_0\right)-2, q\left(x_0\right)+4\right]$. Since $q\left(x_0\right)+2$ is the only point in the interval $\left[q\left(x_0\right)-2, q\left(x_0\right)+4\right]$ which is mapped by $r \circ \sigma_n$ to $a_{m'}$, $q\left(x_1\right) = q\left(x_0\right)+2$.
Case (ii): In this case $m = 2$, $m' = m^+ = 0$, $m'' = 1$, $r\left(\sigma_n\left(q\left(x_0\right)-2\right)\right) = a_1$, $r\left(\sigma_n\left(q\left(x_0\right)+2\right)\right) = a_0$, $r\left(\sigma_n\left(q\left(x_0\right)+4\right)\right) = a_2$, $r\left(\sigma_n\left(q\left(x_0\right)+6\right)\right) = a_1$. Using 3.3 again, we get the result that $f\left(\left\langle x_0, x_1\right\rangle\right) \subset \left[\sigma_n\left(q\left(x_0\right)-2\right), \sigma_n\left(q\left(x_0\right)+6\right)\right]$. Like before, it follows that $x_1 \in L$ and $q\left(x_1\right) \in \left[q\left(x_0\right)-2, q\left(x_0\right)+6\right]$. Since $q\left(x_0\right)+2$

is the only point in the interval $[q(x_0)-2, q(x_0)+6]$ which is mapped by $r\circ\sigma_n$ to $a_{m'}$, $q(x_1) = q(x_0)+2$.
Case (iii): In this case, $f(\langle x_0, x_1\rangle) \subset [\sigma_n(q(x_0)-2), \sigma_n(q(x_0)+2)]$ and the rest of the proof is the same as before.
Case (iv): Suppose $q(x_0) = 6\cdot 2^n$. This cannot occur, because, otherwise, we would have $m=0$, $m'=1$, $m''=2$, $r(\sigma_n(q(x_0)-2)) = a_2$, $r(\sigma_n(q(x_0)+2)) = a_2$. By 3.3 , $f(\langle x_0, x_1\rangle) \subset [\sigma_n(q(x_0)-2), \sigma_n(q(x_0)+2)]$. The last inclusion is a contradiction, because the interval $[q(x_0)-2, q(x_0)+2]$ contains no point which is mapped by $r\circ\sigma_n$ to $a_{m'}$. The case $q(x_0) = 6\cdot 2^{n+1}$ cannot occur either for similar reasons.
Case (v): The proof in this case is very similar to that of in Case (i). By Proposition 2.5, $r(\sigma_n(q(x_0)+2)) = a_{m''}$, $r(\sigma_n(q(x_0)-2)) = a_{m'}$ and $r(\sigma_n(q(x_0)-4)) = a_{m''}$. Since $r(\langle x_0, x_1\rangle) \subset A_{m''}$, Proposition 3.3 implies that neither of the points $\sigma_n(q(x_0)-4)$ and $\sigma_n(q(x_0)+2)$ belongs to $f(\langle x_0, x_1\rangle)$. So $f(\langle x_0, x_1\rangle)$ is contained in $[\sigma_n(q(x_0)-4), \sigma_n(q(x_0)+2)]$. It follows that $x_1 \in L$ and $q(x_1) \in [q(x_0)-4, q(x_0)+2]$. Now, $q(x_1) = q(x_0)-2$, because $q(x_0)-2$ is the only point in the interval $[q(x_0)-2, q(x_0)+4]$ which is mapped by $r\circ\sigma_n$ to $a_{m'}$.
Case (vi): By a similar proof to that of Case (2), we get $q(x_1) = q(x_0)-2$.
Case (vii): Using the same argument, we get that $q(x_1)$ is either $6\cdot 2^{n+1}-2$ or $6\cdot 2^{n+1}+2$ if $q(x_0) = 6\cdot 2^{n+1}$. The lemma holds in both cases, as $g\left(6\cdot 2^{n+1}-2\right) = g\left(6\cdot 2^{n+1}+2\right) = 2$ and $g\left(6\cdot 2^{n+1}\right) = 0$. □

For any integer $j = 0, \dots, 6\cdot 2^{n+2}$, we will adopt the following notation. Let $d(j) = 6\cdot 2^n$ if $|6\cdot 2^n - j| \le \left|6\cdot 2^{n+1} - j\right|$, and let $d(j) = 6\cdot 2^{n+1}$, otherwise. Let $e(j) = 2d(j) - j$.

Proposition 3.9. *$g(e(j)) = g(j)$. On the other hand, if $g(j_0) = g(j)$, $|d(j)-j| < 3\cdot 2^n$ and $|d(j_0)-j_0| < 3\cdot 2^n$, then $d(j_0) = d(j)$ and either $j_0 = j$ or $j_0 = e(j)$.* □

Proposition 3.10. *Suppose $0 \le j \le 6\cdot 2^{n+2}$ and $0 \le g(j) - b \le 6\cdot 2^n$. Let $c' = -1$ if $6\cdot 2^n \le j \le 6\cdot 2^{n+1}$, and $c' = 1$ otherwise. Then, $g(j) - b = g(j - bc')$.* □

Lemma 3.11. *Suppose that the following conditions are satisfied:*

- *$k(\ell) > 0$ for some $\ell = 0, 1, 2$.*
- *h is an integer such that $0 \le h \le k(\ell) - 1$.*
- *j is an even integer such that $0 \le j \le 6\cdot 2^{i+2} - 2$ and $r(\sigma_i(j)) = a_\ell$.*
- *$x_0 \in \sigma_i([j, j+1]) \cap L$ is such that $r(x_0) = t^{(\ell)}_{k(\ell)-h}$.*
- *$q(x_0) \ge 2h + 2$.*
- *$v = {\sigma_i}^{-1}(x_0)$ and $x_1 = \sigma_i(2j - v)$.*

Then $r(x_1) = t^{(\ell)}_{k(\ell)-h}$, $x_1 \in L$ and

1. *$q(x_1)$ is either $q(x_0)$ or $e(q(x_0))$, and*
2. *$q(x_1) = q(x_0)$ if $|d(q(x_0)) - q(x_0)| > 2h$.*

Proof. Observe that x_1 is the only point in $\sigma_i[j-1, j]$ such that $r(x_1) = r(x_0)$. Thus, $r(x_1) = t^{(\ell)}_{k(\ell)-h}$.

We will prove the lemma by induction with respect to h.

For $h = 0$, set $m = \mu(\ell, k(\ell))$ (see 3.2) and use 3.7 to prove that $x_1 \in L_n$ and $q(x_1) = q(x_0)$. So, the proposition is true for $h = 0$.

Now, we will prove the proposition for an arbitrary positive integer h, assuming that the proposition is true when h is replaced by $h - 1$.

Let $x_0' \in \sigma_i([j, j+1])$ and $x_1' \in \sigma_i([j-1, j])$ be such that $r(x_0') = r(x_1') = t^{(\ell)}_{k(\ell)-h+1}$.

Set $m = \mu(\ell, k(\ell) - h)$ and $m' = \mu(\ell, k(\ell) - h + 1)$ and use 3.8 to prove that $x_0' \in L$ and $q(x_0')$ is either $q(x_0) + 2$ or $q(x_0) - 2$. Now, we use the inductive hypothesis to get that $x_1' \in L$ and $q(x_1')$ is either $q(x_0')$ or $e(q(x_0'))$. Consequently,

$$g(q(x_1')) = g(q(x_0')) \tag{3.11.1}$$

Using 3.8 again (with $m = \mu(\ell, k(\ell) - h + 1)$ and $m' = \mu(\ell, k(\ell) - h)$) we get that $x_1 \in L$ and $q(x_1)$ is either $q(x_1') + 2$ or $q(x_1') - 2$. Also $g(q(x_1)) = g(q(x_1')) \pm 2$, $+$ or $-$ depending on which case (c) of 3.8 applies. Use 3.8 again (with the same $m = \mu(\ell, k(\ell) - h + 1)$ and $m' = \mu(\ell, k(\ell) - h)$) to evaluate $g(q(x_0))$ as $g(q(x_0')) \pm 2$. Observe that the same case (c) applies. So, it follows from (3.11.1) that

$$g(q(x_1)) = g(q(x_0)) \tag{3.11.2}$$

Suppose that $|d(q(x_0)) - q(x_0)| > 2h$. Then, $|d(q(x_0')) - q(x_0')| > 2h - 2$. $q(x_1') = q(x_0')$ by the proposition for $h - 1$. Since $q(x_1') = q(x_0')$ is neither $6 \cdot 2^n$ nor $6 \cdot 2^{n+1}$, (3.11.2) implies that $q(x_1) = q(x_0)$.

Finally, suppose that $|d(q(x_0)) - q(x_0)| \leq 2h$. Since $q(x_0')$ is either $q(x_0) + 2$ or $q(x_0) - 2$, $|d(q(x_0')) - q(x_0')| \leq 2h + 2$. Since $q(x_1')$ is either $q(x_0')$ or $e(q(x_0'))$, $|d(q(x_1')) - q(x_1')| \leq 2h + 2$. Lastly, since $q(x_1)$ is either $q(x_1') + 2$ or $q(x_1') - 2$, $|d(q(x_1)) - q(x_1)| \leq 2h + 4$. As $2h + 4 < 3 \cdot 2^n$, 3.9 and (3.11.2) imply that $q(x_1)$ is either $q(x_0)$ or $e(q(x_0))$. $\square$

We will say that f is a rotation on Y if $k(0) > 0$, $k(1) > 0$, $k(2) > 0$ and $\{\mu(0,1), \mu(1,1), \mu(2,1)\} = \{0,1,2\}$. We will say that f is a positive rotation on Y if it is a rotation and $\mu(1,1) = (\mu(0,1))^+$. Similarly, f is a negative rotation on Y if it is a rotation and $\mu(1,1) = (\mu(0,1))^-$.

Lemma 3.12. *Suppose that the following conditions are satisfied:*

- *f is not a rotation on Y.*
- *$k(\ell) > 0$ for some $\ell = 0, 1, 2$.*
- *j_0 and j are even integers such that $0 \leq j < j_0 \leq 6 \cdot 2^{i+2}$ and $r(\sigma_i(j_0)) = r(\sigma_i(j)) = a_\ell$.*
- *$x_0 \in \sigma_i([j_0 - 1, j_0]) \cap L$ and $x_1 \in \sigma_i([j-1, j])$ are such that $r(x_0) = r(x_1) = t^{(\ell)}_1$.*
- *$q(x_0) \geq 6 \cdot 2^{n+1} + 2K + 6$.*

Then, $x_1 \in L$ and $q(x_1) = q(x_0)$.

Proof. Observe that it is enough to prove the lemma only in the case where j is the greatest even integer satisfying the hypothesis. The lemma in its general form will follow by induction. We will consider the following cases.

(i) $j = j_0 - 6$ and either $6 \leq j_0 \leq 6 \cdot 2^i$ or $6 \cdot 2^i + 6 \leq j_0 \leq 6 \cdot 2^{i+1}$ or $6 \cdot 2^{i+1} + 6 \leq j_0 \leq 6 \cdot 2^{i+2}$.
(ii) $j = 2^{i+1} - j_0$ and $2^i < j_0 \leq 2^i + 4$.
(iii) $j = 2^{i+2} - j_0$ and $2^{i+1} < j_0 \leq 2^{i+1} + 4$.

We will outline the proof only in the first case. We will leave proving the remaining cases to the reader.

Let m denote $\mu(\ell,1)$. Let ℓ' and ℓ'' be such that $r(\sigma_i(j_0-2)) = a_{\ell'}$ and $r(\sigma_i(j_0-4)) = a_{\ell''}$. Let $\widetilde{x}_0$ be the only point in the arc $\sigma_i([j,j+1])$ such that $r(\tilde{x}_0) = t_1^{(\ell)}$.

If $k(\ell') > 0$ and, therefore, $t_1^{(\ell')}$ is defined, let $m' = \mu(\ell',1)$, let x_0' be the only point in the arc $\sigma_i([j_0-2,j_0-1])$ such that $r(x_0') = t_1^{(\ell')}$, and let $x_1' \in \sigma_i([j_0-3,j_0-2])$ be such that $r(x_1') = t_1^{(\ell')}$. Similarly, if $t_1^{(\ell'')}$ is defined, let $m'' = \mu(\ell'',1)$, let $x_0'' \in \sigma_i([j_0-4,j_0-3])$ be such that $r(x_0'') = t_1^{(\ell'')}$, and let $x_1'' \in \sigma_i([j_0-5,j_0-4])$ be such that $r(x_1'') = t_1^{(\ell'')}$.

In our proof, we will consider the sequence $S = (x_0, x_0', x_1', x_0'', x_1'', \widetilde{x}_0, x_1)$ after removing all points that are not defined. To each of the consecutive pairs of points in the sequence, we will apply one of 3.7, 3.8 and 3.11. That way, we will be able to claim that all points in the sequence belong to L, and we will be able to evaluate the value of q for each of the points and show that it is greater than $6\cdot 2^{n+1}$. We need to consider several cases depending on the composition of S. For example, $S = (x_0, \widetilde{x}_0, x_1)$ if $k(\ell') = k(\ell'') = 0$. In this case $\langle x_0, \widetilde{x}_0\rangle \subset A_{m^+} \cap A_{m^-}$. By 3.7, $\widetilde{x}_0 \in L$ and $q(\widetilde{x}_0) = q(x_0)$. Now, by 3.11 applied to $\widetilde{x}_0$ and x_1 with $h = k(\ell)-1$, we get $x_1 \in L$ and $q(x_1) = q(\widetilde{x}_0)$.

If $S = (x_0, x_0', x_1', x_0'', x_1'', \widetilde{x}_0, x_1)$, we use 3.11 three times. First, we use 3.11(2) (applied to x_0' and x_1' with $h = k(\ell')-1$) to get $q(x_1') = q(x_0')$. Then, we use 3.11(2) (applied to x_0'' and x_1'' with $h = k(\ell'')-1$) to get $q(x_1'') = q(x_0'')$. Finally, we use 3.11(2) (applied to $\widetilde{x}_0$ and x_1 with $h = k(\ell)-1$) to get $q(x_1) = q(\widetilde{x}_0)$.

To evaluate $q(x_0')$ in terms of $q(x_0)$ we apply either 3.7 or 3.8. If $m' = m$, 3.7 implies that $q(x_0') = q(x_0)$. If $m' \neq m$, $q(x_0') = q(x_0) \pm 2$ by 3.8, $+$ or $-$ depending on which case of 3.8 applies. Similarly, $q(x_0'') = q(x_1')$ if $m'' = m'$, and, otherwise, $q(x_0'') = q(x_1') \pm 2$. Finally, $q(\widetilde{x}_0) = q(x_1'')$ if $m = m''$, and, otherwise, $q(\tilde{x}_0) = q(x_1'') \pm 2$.

Since f is not a rotation on Y, $\{m, m', m''\} \neq \{0,1,2\}$. If $m' = m'' = m$, then 3.7 is used three times, and as the result we get $q(x_1) = q(x_0)$. If the set $\{m, m', m''\}$ has exactly two elements, then 3.7 is used once and 3.8 is used twice. Observe that if in the first use of 3.8 we have $q(\cdot) = q(\cdot) \pm 2$, that in the second use of 3.8 we have $g(q(\cdot)) = g(q(\cdot)) \mp 2$. So, the two instances of 3.8 cancel each other and we also get $q(x_1) = q(x_0)$.

The argument in each of the remaining cases $S = (x_0, x_0', x_1', \widetilde{x}_0, x_1)$ and $S = (x_0, x_0'', x_1'', \widetilde{x}_0, x_1)$ is similar to the one presented above and it will be omitted. □

Recall that $f(u) = p_n = \sigma_n\left(6\cdot 2^{n+2}\right)$ and $f(w) = p = \sigma_n(-1)$. Let u_0' be the first point in the arc $L = \langle u, w\rangle$ such that $f(u_0') = \sigma_n\left(6\cdot 2^{n+2}-4\right)$. By 3.4, $r(\langle u, u_0'\rangle) \cap T \neq \emptyset$. Let $u_0 \in \langle u, u_0'\rangle$ be such that $r(\langle u, u_0\rangle) \cap T = r(u_0)$. Since $f(\langle u, u_0'\rangle) \subset \sigma_n\left(\left[6\cdot 2^{n+2}-4, 6\cdot 2^{n+2}\right]\right)$,

$$q(u_0) \geq 6\cdot 2^{n+2} - 4 \tag{$\bullet u_0$}$$

There is an even integer j_1 such that $0 \leq j_1 \leq 6\cdot 2^{i+2}$ and $u_0 \in \sigma_i([j_1-1, j_1^*])$ where j_1^* is the minimum of j_1+1 and $6\cdot 2^{i+2}$. Let $\ell_1 = 0,1,2$ be such that $r(\sigma_i(j_1)) = a_{\ell_1}$ and let $u_1 \in \sigma_i([j_1-1, j_1])$ be the such that $r(u_1) = t_1^{(\ell_1)}$.

We will now observe that $u_1 \in L$ and estimate $q(u_1)$. Let h' be such that $r(u_0) = t_{h'}^{(\ell_1)}$. For each integer $\tau = 1, \ldots, h'$, let $u_{1,\tau} \in \sigma_i([j_1-1, j_1])$ be such that $r(u_{1,\tau}) =$

$t_\tau^{(\ell_1)}$. Either $u_{1,h'} = u_0$ or 3.11 implies that $u_{1,h'} \in L$ and $q(u_{1,h'}) = q(u_0)$. Now, we use 3.8 $h' - 1$ times to get that $u_{1,\tau} \in L$ and $q(u_{1,\tau}) \geq q(u_0) - 2(h' - \tau)$ for each $\tau = h' - 1, \ldots, 1$. As $u_{1,1} = u_1$ and $h' \leq K$, $(\bullet u_0)$ implies

$$(\bullet u_1) \qquad q(u_1) \geq 6 \cdot 2^{n+2} - 2K - 2$$

Let u_2 be the last point in the arc $\langle u_1, w\rangle$ such that there is an even integer $j_2 = 0, \ldots, 6 \cdot 2^{i+2}$ and there is $\ell_2 = 0, 1, 2$ such that $u_2 \in \sigma_i([j_2 - 1, j_2])$, $r(\sigma_i(j_2)) = a_{\ell_2}$, $k(\ell_2) > 0$, $r(u_2) = t_1^{(\ell_2)}$.

Let u_3' be the first point in the arc $L = \langle w, u\rangle$ such that $f(u_3') = \sigma_n(4)$. By 3.4, $r(\langle w, u_3'\rangle) \cap T \neq \emptyset$. Let $u_3 \in \langle w, u_3'\rangle$ be such that $r(\langle w, u_3\rangle) \cap T = r(u_3)$. Since $f(\langle w, u_3'\rangle) \subset \sigma_n([-1, 4])$,

$$(\bullet u_3) \qquad q(u_3) \leq 4$$

There is an even integer j_3 such that $0 \leq j_3 \leq 6 \cdot 2^{i+2}$ and $u_3 \in \sigma_i([j_3 - 1, j_3 + 1])$. Let $\ell_3 = 0, 1, 2$ be such that $r(\sigma_i(j_3)) = a_{\ell_3}$.

We will now estimate $q(u_2)$. If $j_2 = j_3$, then $u_2 = u_3$ and $q(u_2) \leq 4$. Suppose then that $j_2 > j_3$. Set $u_{3,0} = u_2$. For each $\tau = 1, \ldots, k(\ell_3)$, let $u_{3,\tau} \in \sigma_i([j_3, j_3 + 1])$ be such that $r(u_{3,\tau}) = t_\tau^{(\ell_3)}$. There is an integer $h'' = 1, \ldots, k(\ell_3)$ such that $r(u_3) = r(u_{3,h''})$. Either $u_{3,h''} = u_3$ or 3.11 implies that $q(u_{3,h''}) = q(u_3)$. Now, we use either 3.7 or 3.8 to get the result that $q(u_{3,\tau}) \geq q(u_{3,\tau-1}) - 2$ for $\tau = 1, \ldots, h''$. Combining those inequalities with $(\bullet u_3)$ we get

$$(\bullet u_2) \qquad q(u_2) \leq 2K + 4$$

Lemma 3.13. *f is a rotation on Y.*

Proof. Let $j \geq j_2$ be the least even integer such that $r(\sigma_i(j)) = a_{\ell_1}$. Let $u_2' \in \sigma_i([j - 1, j])$ be the such that $r(u_2') = t_1^{(\ell_1)}$. By 2.3, $j \leq j_2 + 6$. Using 3.8 and 3.11 we get the result that $q(u_2') \leq q(u_2) + 6$. So,

$$(3.13.1) \qquad q(u_2') \leq 2K + 10$$

by $(\bullet u_2)$.

Suppose that f is not a rotation on Y. Then, it follows from 3.12 that $q(u_2') = q(u_1)$. Since $q(u_1) \geq 6 \cdot 2^{n+2} - 2K - 2$ (see $(\bullet u_1)$), $q(u_2') \geq 6 \cdot 2^{n+2} - 2K - 2$, which contradicts (3.13.1) and the choice of ν. □

Since f is a rotation on Y, $k(0) > 0$, $k(1) > 0$ and $k(2) > 0$. Set $c_f = 1$ if f is a positive rotation on Y, and set $c_f = -1$, otherwise.

For an integer $j = 0, \ldots, 6 \cdot 2^{n+2}$, let

$$\gamma(j) = \begin{cases} 0, & \text{if } 0 \leq j \leq 6 \cdot 2^n, \\ 1, & \text{if } 6 \cdot 2^n < j \leq 6 \cdot 2^{n+1}, \\ 2, & \text{if } 6 \cdot 2^{n+1} < j \leq 6 \cdot 2^{n+2}. \end{cases}$$

Observe that the values of $g(j)$ and $\gamma(j)$ determine j.

For any set $S \subset \{0, 1, 2\}$ and any integer $j = 0, \ldots, 6 \cdot 2^n$, let

$$\alpha(j, S) = \begin{cases} S \cup \{0, 1\}, & \text{if } S \cap \{0, 1\} \neq \emptyset \text{ and } 6 \cdot 2^n - 2K \leq j \leq 6 \cdot 2^n, \\ S \cup \{1, 2\}, & \text{if } S \cap \{1, 2\} \neq \emptyset \text{ and } 0 \leq j \leq 2K, \\ S, & \text{otherwise.} \end{cases}$$

For any integers $j_0, j = 0, \ldots, 6 \cdot 2^n$ and set $S \subset \{0, 1, 2\}$, let $\alpha(j, j_0, S)$ denote the set $\alpha(j, \alpha(j_0, S))$. If S is a singleton $\{m\}$, we will simply write $\alpha(j_0, m)$ and $\alpha(j, j_0, m)$ instead of $\alpha(j_0, \{m\})$ and $\alpha(j, j_0, \{m\})$, respectively.

Lemma 3.14. *Suppose that the following conditions are satisfied:*

(i) j_0 *and* j *are even integers such that either* $0 \leq j < j_0 \leq 6 \cdot 2^i$ *or* $6 \cdot 2^i \leq j < j_0 \leq 6 \cdot 2^{i+1}$ *or* $6 \cdot 2^{i+1} \leq j < j_0 \leq 6 \cdot 2^{i+2}$.
(ii) $c = -1$ *if* $6 \cdot 2^i \leq j < j_0 \leq 6 \cdot 2^{i+1}$, *and* $c = 1$ *otherwise.*
(iii) $r(\sigma_i(j_0)) = a_{\ell_0}$ *and* $r(\sigma_i(j)) = a_\ell$ *for some* $\ell_0, \ell = 0, 1, 2$.
(iv) $x_0 \in \sigma_i([j_0 - 1, j_0]) \cap L$ *is such that* $r(x_0) = t_1^{(\ell_0)}$.
(v) $x \in \sigma_i([j-1, j])$ *is such that* $r(x) = t_1^{(\ell)}$.
(vi) *Either* $q(x_0) > 2^n + 2K$ *or* $\min(g(q(x_0)), g(q(x_0)) - cc_f(j_0 - j)) > 2K$.

Then $x \in L$,

(1) $g(q(x)) = g(q(x_0)) - cc_f(j_0 - j)$,
(2) $\gamma(q(x)) \in \alpha(g(q(x)), g(q(x_0)) - 2cc_f, \gamma(q(x_0)))$,
(3) $q(x) \geq 6 \cdot 2^{n+1} - 2K$ *if* $q(x_0) \geq 6 \cdot 2^{n+1} + 2K$, *and*
(4) $q(x) \geq 6 \cdot 2^n - 2K$ *if* $q(x_0) \geq 6 \cdot 2^n + 2K$.

Proof. We will first observe that (3) and (4), follow from (1) and (2). Suppose $q(x_0) \geq 6 \cdot 2^{n+1} + 2K$. Then $\gamma(q(x_0)) = 2$ and $g(q(x_0)) \geq 2K$. If $cc_f = -1$, then $g(q(x_0)) - 2cc_f > 2K$ and $g(q(x)) = g(q(x_0)) - cc_f(j_0 - j) > 2K$. Now, it follows from (2) that $\gamma(q(x)) = 2$, so (3) is true if $cc_f = -1$. Suppose $cc_f = 1$. Then, $g(q(x)) = g(q(x_0)) - cc_f(j_0 - j) \leq g(q(x_0)) - 2cc_f$. If $g(q(x)) > 2K$, (2) implies that $\gamma(q(x)) = 2$ and (3) is true again. If, on the other hand, $g(q(x)) \leq 2K$, then $\gamma(q(x)) \in \{1, 2\}$, $6 \cdot 2^{n+1} - 2K \leq q(x) \leq 6 \cdot 2^{n+1} + 2K$ and the proof of (3) is complete. The proof of (4) is essentially the same and will be omitted.

We will prove conditions (1) and (2) of the lemma by induction with respect to $j_0 - j$.

We start with $j_0 - j = 2$. There exists exactly one point $x_1 \in \sigma_i([j, j+1])$ is such that $r(x_1) = t_1^{(\ell)}$. Let $m = \mu(\ell_0, 1)$ and $m' = \mu(\ell, 1)$. Observe that x_0 and x_1 satisfy the hypothesis of 3.8. We can now use 3.8 to prove that $x_1 \in L$ and to evaluate $g(q(x_1))$ as $g(q(x_0)) \pm 2$, $+$ if $m' = m^+$, and $-$, otherwise. We need to consider four cases depending on the values of c_f and c.
Case $c_f = 1$ and $c = 1$: In this case $\ell = {\ell_0}^-$ and $m' = \mu(\ell, 1) = \mu({\ell_0}^-, 1) = (\mu(\ell_0, 1))^- = m^-$. So, $g(q(x_1)) = g(q(x_0)) - 2 = g(q(x_0)) - 2cc_f$.
Case $c_f = -1$ and $c = 1$: In this case again $\ell = {\ell_0}^-$. But, $m' = \mu(\ell, 1) = \mu({\ell_0}^-, 1) = (\mu(\ell_0, 1))^+ = m^+$. So, $g(q(x_1)) = g(q(x_0)) + 2 = g(q(x_0)) - 2cc_f$.
Case $c_f = 1$ and $c = -1$: In this case $\ell = {\ell_0}^+$ and $m' = \mu(\ell, 1) = \mu({\ell_0}^+, 1) = (\mu(\ell_0, 1))^+ = m^+$. So, $g(q(x_1)) = g(q(x_0)) + 2 = g(q(x_0)) - 2cc_f$.
Case $c_f = -1$ and $c = -1$: In this case $\ell = {\ell_0}^+$ and $m' = \mu(\ell, 1) = \mu({\ell_0}^+, 1) = (\mu(\ell_0, 1))^- = m^-$. So, $g(q(x_1)) = g(q(x_0)) - 2 = g(q(x_0)) - 2cc_f$.
We proved that $g(q(x_1)) = g(q(x_0)) - 2cc_f$ in all of the cases. Using 3.11 and 3.9, we prove that $x \in L$ and $g(q(x)) = g(q(x_1))$. Thus, $g(q(x)) = g(q(x_0)) - 2cc_f$.

To prove 3.14(2) in the case $j = j_0 - 2$, we need to observe when $\gamma(q(x_0))$, $\gamma(q(x_1))$ and $\gamma(q(x))$ could be different. If $\gamma(q(x_1)) \neq \gamma(q(x_0))$ then either $\{\gamma(q(x_0)), \gamma(q(x_1))\} = \{0, 1\}$ and $g(q(x_1)) \geq 6 \cdot 2^n - 2$, or $\{\gamma(q(x_0)), \gamma(q(x_1))\} = \{1, 2\}$ and $g(q(x_1)) \leq 2$. If $\gamma(q(x)) \neq \gamma(q(x_1))$ then either $\{\gamma(q(x_1)), \gamma(q(x))\} = \{0, 1\}$ and $g(q(x_1)) \geq 6 \cdot 2^n - 2K$, or $\{\gamma(q(x_1)), \gamma(q(x))\} = \{1, 2\}$ and $g(q(x_1)) \leq 2K$. 3.14(2) for $j_0 - j = 2$ readily follows from the last two sentences.

Suppose that the proposition is true if j is replaced by $j+2$, and prove it for j. Suppose also that the conditions (i)–(vi) are satisfied for j. Let $\ell' = 0, 1, 2$ be such that $\sigma_i(j+2) = a_{\ell'}$, and let $x' \in \sigma_i([j+1, j+2])$ be such that $r(x') = t_1^{(\ell')}$.

We will infer the proposition for j_0, j, x_0 and x from the inductive hypothesis by applying it twice, first to j_0, $j+2$, x_0 and x', and then to $j+2$, j, x' and x. In order to be able to do so, we need to verify the conditions (i), (ii) and (vi) in the both cases.

Since $j < j+2 \leq j_0$, the conditions (i) and (ii) are satisfied by all three pairs (j_0, j), $(j_0, j+2)$ and $(j+2, j)$ with the same c.

Proof of condition (vi) for j_0, $j+2$, x_0 and x': We may assume that $q(x_0) \leq 2^n + 2K$. Since $g(q(x_0)) > 2K$ by (vi) for j_0, j, x_0 and x, we need to observe that $g(q(x_0)) - cc_f(j_0 - j - 2) > 2K$. If $cc_f = -1$, $g(q(x_0)) - cc_f(j_0 - j - 2) > g(q(x_0)) > 2K$. On the other hand, if $cc_f = 1$, $g(q(x_0)) - cc_f(j_0 - j - 2) > g(q(x_0)) - cc_f(j_0 - j)$ and the last number is greater that $2K$ by (vi) for j_0, j, x_0 and x.

We may now use the proposition for j_0, $j+2$, x_0 and x' and infer that, $x' \in L$,

(1′) $g(q(x')) = g(q(x_0)) - cc_f(j_0 - j - 2)$, and

(2′) $\gamma(q(x')) \in \alpha(g(q(x')), g(q(x_0)) - 2cc_f, \gamma(q(x_0)))$.

Proof of condition (vi) for $j+2$, j, x' and x: Suppose $q(x_0) > 2^n + 2K$. In that case, either $q(x') > 2^n + 2K$ or $g(q(x')) \geq 2^n - 2K$. In this last case, both $g(q(x'))$ and $g(q(x')) - 2cc_f$ are greater than $2K$. So, condition (vi) is satisfied if $q(x_0) > 2^n + 2K$. Therefore, we may assume that $q(x_0) \leq 2^n + 2K$, and $\min(g(q(x_0)), g(q(x_0)) - cc_f(j_0 - j)) > 2K$. It follows that $g(q(x')) = g(q(x_0)) - cc_f(j_0 - j - 2)$ is also greater than $2K$. As $g(q(x')) - 2cc_f = g(q(x_0)) - cc_f(j_0 - j)$, condition (vi) is satisfied for $j+2$, j, x' and x.

We may now use the proposition for $j+2$, j, x' and x and infer that, $x \in L$,

(1″) $g(q(x)) = g(q(x')) - 2cc_f$, and

(2″) $\gamma(q(x)) \in \alpha(g(q(x)), g(q(x')) - 2cc_f, \gamma(q(x'))) = \alpha(g(q(x)), \gamma(q(x')))$.

Observe that 3.14 (1) follows readily from (1′) and (1″). To complete the proof of the proposition we need to show 3.14 (2).

Denote the set $\alpha(g(q(x_0)) - 2cc_f, \gamma(q(x_0)))$ by S. We need to show that $\gamma(q(x)) \in \alpha(g(q(x)), S)$.

Observe that

$$\alpha(g(q(x_0)) - 2cc_f, S) = S.$$

We will consider two cases depending on whether $\gamma(q(x'))$ belongs to S.

Case $\gamma(q(x')) \notin S$: In this case, $\gamma(q(x')) \notin \alpha(g(q(x_0)) - 2cc_f, S)$. On the other hand, $\gamma(q(x')) \in \alpha(g(q(x')), S) = \alpha(g(q(x')), g(q(x_0)) - 2cc_f, \gamma(q(x_0)))$ by (2′). It follows that either

- $g(q(x_0)) - 2cc_f < 6 \cdot 2^n - 2K$ and $g(q(x')) \geq 6 \cdot 2^n - 2K$, or
- $g(q(x_0)) - 2cc_f > 2K$ and $g(q(x')) \leq 2K$.

Since $g(q(x'))$ is between $g(q(x_0)) - 2cc_f$ and $g(q(x))$, we have the result that either $g(q(x')), g(q(x)) \in [6 \cdot 2^n - 2K, 6 \cdot 2^n]$ or $g(q(x')), g(q(x)) \in [0, 2K]$. This implies that $\alpha(g(q(x')), S) = \alpha(g(q(x)), S)$ and

$$\alpha(g(q(x)), \alpha(g(q(x')), S)) = \alpha(g(q(x)), S) \tag{3.14.1}$$

Since $\gamma(q(x')) \in \alpha(g(q(x')), S)$, it follows from (3.14.1) and (2″) that $\gamma(q(x)) \in \alpha(g(q(x)), \gamma(q(x'))) \subset \alpha(g(q(x)), \alpha(g(q(x')), S)) = \alpha(g(q(x)), S)$.

Thus 3.14 (2) is true in when $\gamma(q(x')) \notin S$.

Case $\gamma(q(x')) \in S$: $\gamma(q(x)) \in \alpha(g(q(x)), \gamma(q(x'))) \subset \alpha(g(q(x)), S)$. $\square$

Let $y_0 \in \sigma_i\left(\left[6 \cdot 2^i - 1, 6 \cdot 2^i\right]\right)$ and $y_1 \in \sigma_i\left(\left[6 \cdot 2^{i+1} - 1, 6 \cdot 2^{i+1}\right]\right)$ be such that $r(y_0) = r(y_1) = t_1^{(0)}$.

Let j_1, u_1, j_2 and u_2 be as defined before Lemma 3.13. Recall that $q(u_1) \geq 6 \cdot 2^{n+2} - 2K - 2$ by ($\bullet u_1$). Consequently, $q(u_1) > 6 \cdot 2^{n+1} + 2K$ by the choice of ν.

Lemma 3.15. $6 \cdot 2^{i+1} < j_1 \leq 6 \cdot 2^{i+2}$.

Proof. If the lemma is not true, then either $j_1 \leq 6 \cdot 2^i$ or $6 \cdot 2^i < j_1 \leq 6 \cdot 2^{i+1}$.

Case $j_1 \leq 6 \cdot 2^i$: Use 3.14 with $j_0 = j_1$ and $j = j_2$ to get that $q(u_2) \geq 6 \cdot 2^{n+1} - 2K$, which contradicts ($\bullet u_2$) and the choice of ν.

Case $6 \cdot 2^i < j_1 \leq 6 \cdot 2^{i+1}$: Use 3.14 with $j_0 = j_1$ and $j = 6 \cdot 2^i$ to get that $y_0 \in L$ and $q(y_0) \geq 6 \cdot 2^{n+1} - 2K > 6 \cdot 2^n + 2K$. Since $y_0 \in L$, the choice of j_2 and ($\bullet u_2$) imply that $j_2 \leq 6 \cdot 2^i$. Now, use 3.14 with $j_0 = 6 \cdot 2^i$ and $j = j_2$ to get that $q(u_2) \leq 6 \cdot 2^n - 2K$, which again contradicts ($\bullet u_2$) and the choice of ν. $\square$

Lemma 3.16. *$y_0, y_1 \in L$,*

(1) $6 \cdot 2^n - 2K \leq q(y_0) < 6 \cdot 2^n + 2K$ *and*
(2) $6 \cdot 2^{n+1} - 2K \leq q(y_1) < 6 \cdot 2^{n+1} + 2K$.

Proof. Use 3.14 with $j_0 = j_1$ and $j = 6 \cdot 2^{i+1}$ to get that $y_1 \in L$ and $q(y_1) \geq 6 \cdot 2^{n+1} - 2K$. Consequently, $q(y_1) > 6 \cdot 2^n + 2K$ by the choice of ν.

Use 3.14 again, this time with $j_0 = 6 \cdot 2^{i+1}$ and $j = 6 \cdot 2^i$ to get that $y_0 \in L$ and $q(y_0) \geq 6 \cdot 2^n - 2K$.

Suppose $q(y_0) \geq 6 \cdot 2^n + 2K$. In that case, use 3.14 yet again, this time with $j_0 = 6 \cdot 2^i$ and $j = j_2$ to get that and $q(u_2) \geq 6 \cdot 2^n - 2K$, which contradicts ($\bullet u_2$) and the choice of ν. Thus (1) is true.

Now, suppose $q(y_1) \geq 6 \cdot 2^{n+1} + 2K$. In that case our earlier use 3.14 with $j_0 = 6 \cdot 2^{i+1}$ and $j = 6 \cdot 2^i$ would yield $q(y_0) \geq 6 \cdot 2^{n+1} - 2K > 6 \cdot 2^n + 2K$, a contradiction with previously proven (1). Thus, (2) is true and the proof of the lemma is complete. $\square$

Lemma 3.17. *f is a positive rotation on Y.*

Proof. Recall that $q(u_1) \geq 6 \cdot 2^{n+2} - 2K - 2$ by ($\bullet u_1$). Consequently, $g(q(u_1)) \geq 6 \cdot 2^n - 2K - 2$ and $\gamma(q(u_1)) = 2$.

Suppose that the lemma is not true. In view of 3.13, we may assume that f is a negative rotation on Y. Then, $c_f = -1$.

Use 3.14 with $j_0 = j_1$ and $j = 6 \cdot 2^{i+1}$ to get that $g(q(y_1)) = g(q(u_1)) + j_1 - 6 \cdot 2^{i+1}$ and $\gamma(q(y_1)) \in \alpha(g(q(y_1)), g(q(u_1)) + 2, 2)$. As $j_1 > 6 \cdot 2^{i+1}$, $g(q(y_1)) > g(q(u_1)) \geq 6 \cdot 2^n - 2K - 2$. It follows that $\alpha(g(q(y_1)), g(q(u_1)) + 2, 2) = \{2\}$. Thus, $\gamma(q(y_1)) = 2$ and $q(y_1) \geq 6 \cdot 2^{n+2} - 2K - 2$, which contradicts 3.16 (2) and the choice of ν. $\square$

Since f is a positive rotation on Y, $c_f = 1$. Using 3.14 with $j_0 = 6 \cdot 2^{i+1}$ and $j = 6 \cdot 2^i$ we now get that $g(q(y_0)) = g(q(y_1)) + 6 \cdot 2^{i+1} - 6 \cdot 2^i$. Thus

$$g(q(y_0)) = g(q(y_1)) + 6 \cdot 2^i \tag{*}$$

It follows from 3.16 (1) that $6 \cdot 2^n - 2K \leq g(q(y_0)) \leq 6 \cdot 2^n$. Combining this result with (*), we get

$$6 \cdot 2^n - 2K - g(q(y_1)) \leq 6 \cdot 2^i \leq 6 \cdot 2^n - g(q(y_1)) \tag{**}$$

Since $0 \leq g(q(y_1)) \leq 2K$ by 3.16 (2), (**) implies

$$(***) \qquad 6 \cdot 2^n - 4K \leq 6 \cdot 2^i \leq 6 \cdot 2^n$$

Clearly, $i \leq n$. On the other hand, by the choice of ν we have $2^{n-1} \geq 2^{\nu-1} > K$. Thus $6 \cdot 2^n - 4K > 6 \cdot 2^n - 6K > 6 \cdot 2^n - 6 \cdot 2^{n-1} = 6 \cdot 2^{n-1}$. So, (***) implies $2^{n-1} < 2^i$. As i and n are integers, $i \geq n$. Consequently, $n = i$ and the proof of the theorem is complete. □

4. An uncountable collection of incomparable retracts of D

Each positive integer can be uniquely represented in the form

$$n = 3^i (3k + j)$$

where $i, k = 0, 1, \ldots$ and j is either 1 or 2. We will call this representation the ternary representation of n. Let $\mathcal{E}$ denote the set of sequences $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots)$ with values in the set $\{0, 1\}$. For each $\varepsilon \in \mathcal{E}$ we will define a set $N[\varepsilon] \subset \{1, 2, \ldots\}$ in the following way. In n is a positive integer with ternary representation $3^i(3k + j)$ is in $N[\varepsilon]$ if and only if either $\varepsilon_k = 0$ and $j = 1$ or $\varepsilon_k = 1$ and $j = 2$. Set

$$D[\varepsilon] = Y \cup \bigcup_{n \in N[\varepsilon]} L_n$$

Let $r[\varepsilon] : D \to D[\varepsilon]$ be defined by

$$r[\varepsilon](x) \begin{cases} x, & \text{if } x \in D[\varepsilon], \\ r(x), & \text{otherwise.} \end{cases}$$

Observe that $r[\varepsilon]$ is a continuous retraction of D onto $D[\varepsilon]$.

Theorem 4.1. *Suppose $\varepsilon, \eta \in \mathcal{E}$, $\varepsilon \neq \eta$, $X \subset D[\eta]$ is a continuum, and $f : X \to D$ is a mapping. Then, $D[\varepsilon] \setminus f(X) \neq \emptyset$.*

Proof. Let $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots)$ and $\eta = (\eta_0, \eta_1, \ldots)$. Since $\varepsilon \neq \eta$, there $k = 0, 1, \ldots$ such that $\varepsilon_k \neq \eta_k$. Set $j = 1$ if $\varepsilon_k = 0$, and set $j = 2$, otherwise. Let ν be as in Theorem 3.1. Take an integer i such that $n = 3^i(3k + j) \geq \nu$. It follows from the choice of j that $L_n \subset D[\varepsilon]$ and $L_n \cap D[\eta] = \{p\}$. By Theorem 3.1, $f^{-1}(p_n) = \emptyset$. Thus $p_n \in D[\varepsilon] \setminus f(X)$. □

Corollary 4.2. *$\mathcal{D} = \{D[\varepsilon]\}_{\varepsilon \in \mathcal{E}}$ is a collection of $2^{\aleph_0}$ dendroids (fans) such that each two members of $\mathcal{D}$ are incomparable by continuous functions.*

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