A continuum X whose hyperspace of subcontinua is not its continuous image

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Abstract

We construct a metric continuum X such that the hyperspace of subcontinua, $C(X)$, of X is not a continuous image of X. This answers a question by I. Krzemińska and J. R. Prajs.

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INTRODUCTION

A *continuum* is a nonempty, nondegenerate compact connected metric space. Given a continuum X, we consider the following hyperspaces of X.

$$2^X = \{A \subset X : A \text{ is closed and nonempty}\},$$
$$C(X) = \{A \in 2^X : A \text{ is connected}\}.$$

Both are endowed with the Hausdorff metric H.

The problem of determining conditions for the existence of onto mappings between X, $C(X)$ and 2^X has been considered in [1], [2] and [3]. In [2, 3.6] it was proved that there is always a mapping from 2^X onto $C(X)$. In [2, 3.5], was also proved that there is a mapping from X onto 2^X if and only if (a) X is locally connected or (b) X contains an open set with countably many components. Thus, if X satisfies (a) or (b), then $C(X)$ is a continuous image of X. For a more detailed discussion on this topic see Chapter IV of [3]. In [1, Question 2], I. Krzemińska and J. R. Prajs asked if there exists a continuum X with no continuous surjection from X onto $C(X)$ (this question also was posted on the Open Problems on Continuum Theory website of J. R. Prajs and W. J. Charatonik). Here, we answered this question in the negative by constructing a continuum X such that $C(X)$ is not a continuous image of X. The related questions: Is $C(C(X))$ a continuous image of $C(X)$ for each continuum X? ([1, Question 3]) and; if Y is a continuous image of X, is $C(Y)$ a continuous image of $C(X)$? ([1, Question 1]) remain open.

THE EXAMPLE

Let Z be the $\sin(\frac{1}{x})$-continuum defined as the closure in the plane of the graph of the function $\sin(\frac{1}{x})$ with the interval $(0,1]$ as its domain. Then Z is the union of the limit arc $J = \{0\} \times [-1,1]$ and the ray $T = Z - J$. Consider the continuum Y obtained by identifying the points $(0,-1)$ and $(0,1)$ in Z and let $h : Z \to Y$ be the quotient map. Then Y is the union of the simple closed curve $C = h(J)$ and the ray $h(T)$. The continuum Y constructed in this way is going to be called a *packman* having its *limit circle* C, its *ray* $h(T)$ and its *peak point* $h((0,1))$.

Consider the circle $S = \{(x,y,0) \in E^3 : x^2 + y^2 = 1\}$ in the Euclidean space E^3. Fix a sequence $\{z_n\}_{n=1}^{\infty}$ in S and a point $z_0 \in S$ such that $\lim z_n = z_0$ and the points $z_0, z_1, z_2, \ldots$ are pairwise different.

Consider a sequence of packman continua $\{Y_n\}_{n=1}^{\infty}$ in E^3 with the following properties:

(a) S is the limit circle of Y_n for each $n \in \mathbb{N}$,
(b) z_n is the peak point of Y_n for each $n \in \mathbb{N}$,
(c) the rays $R_1 = Y_1 - S$, $R_2 = Y_2 - S, \ldots$ are pairwise disjoint,
(d) $\lim Y_n = S$.

The continuum X is then defined as $X = Y_1 \cup Y_2 \cup \ldots$.

We can assume also that X satisfies the following property:

(e) given a path $\alpha : [0,1] \to S$ and $\lambda > 0$, there exists $\delta > 0$ such that, if $z_n \notin \alpha([0,1])$, $q \in R_n$ and $\|\alpha(0) - q\| < \delta$, then there exists a path $\beta : [0,1] \to R_n$ such that $\beta(0) = q$ and $\|\alpha(t) - \beta(t)\| < \lambda$ for each $t \in [0,1]$, Observe that X satisfies the following properties:

(f) if J is an arc in S, $z_n \in J$ and z_n is not an end point of J, then $J \notin \mathrm{cl}_{C(X)}(C(R_n))$, where $\mathrm{cl}_{C(X)}(C(R_n)) = \{A \in C(X) : A \subset R_n\}$,

(g) if J is an arc in $S - \{z_n\}$, then $J \in \mathrm{cl}_{C(X)}(C(R_n))$,

(h) S is a terminal subcontinuum of X, that is, if $A \in C(X)$ and $A \cap S \neq \emptyset$, then $A \subset S$ or $S \subset A$.

We are going to prove that there is no a continuous map form X onto $C(X)$, By way of contradiction, suppose that there exists a continuous onto map $f : X \to C(X)$.

Given $n \in \mathbb{N}$ and a continuum $J \subset S - \{z_n\}$, by Property (g), there exists a sequence of continua $\{J_m\}_{m=1}^{\infty}$ such that $J_m \subset R_n$ for each $m \in \mathbb{N}$ and $\lim J_m = J$. Since f is onto, for each $m \in \mathbb{N}$, there exists $p_m \in X$ such that $f(p_m) = J_m$. Since X is compact, we can assume that $\lim p_m = p_0$, for some $p_0 \in X$. By continuity of f, $f(p_0) = J$. We need to prove the following.

Claim 1. If $\{B_k\}_{k=1}^{\infty}$ is a sequence in $C(X)$ such that $p_0 \in B_k$ for each $k \in \mathbb{N}$ and $\lim B_k = \{p_0\}$, then there exists $k \in \mathbb{N}$ such that $B_k \cap \{p_1, p_2, \ldots\} = \emptyset$.

In order to prove Claim 1, for each $k \in \mathbb{N}$, let $C_k = \bigcup\{f(p) : p \in B_k\}$. It is easy to show that (see [3, Lemmas 1.43 and 1.48]) $J \subset C_k \in C(X)$ for each $k \in \mathbb{N}$ and $\lim C_k = J$. Since $z_n \notin J$, there exists $k \in \mathbb{N}$ such that $z_n \notin C_k$. By Property (h), $C_k \subset S$. Thus $C_k \cap R_n = \emptyset$ and C_k does not contain any set J_m. Hence, $B_k \cap \{p_1, p_2, \ldots\} = \emptyset$.

As a consequence of Claim 1, we obtain that X is not locally connected at p_0. Thus $p_0 \in S$. Another consequence of Claim 1 is that S cannot contain infinitely many elements of the sequence $\{p_m\}_{m=1}^{\infty}$.

In particular, we have that for each arc $J \subset S - \{z_1\}$, there exists a point $p_0 \in S$ such that $f(p_0) = S$. Taking a sequence of arcs in $S - \{z_1\}$ converging to S, we obtain that there exists a point $p \in S$ such that $f(p) = S$. Let $K = f^{-1}(\{S\})$. Hence, $K \cap S \neq \emptyset$.

Since the set $F = \{f(z_n) \in C(X) : n \in \{0, 1, 2, \ldots\}\}$ is countable, we can choose a point $q_0 \in S$ such that $\{q_0\} \notin F$.

For each $n \in \mathbb{N}$, fix a sequence of points $\{q_m^{(n)}\}_{m=1}^{\infty}$ in R_n such that $\lim q_m^{(n)} = q_0$. Since f is onto, for each $m \in \mathbb{N}$, there exists a point $p_m^{(n)} \in X$ such that $f(p_m^{(n)}) =$

$\{q_m^{(n)}\}$. Since X is compact, we may assume that the sequence $\{p_m^{(n)}\}_{m=1}^{\infty}$ converges to a point $w_n \in X$. By the considerations we made before, $w_n \in S$ and we may assume that $p_m^{(n)} \in X - S$ for each $m \in \mathbb{N}$. If there exists $k \in \mathbb{N}$ such that R_k contains infinitely many elements of the sequence $\{p_m^{(n)}\}_{m=1}^{\infty}$, we choose one k_n with such a property and, taking a subsequence if necessary, we assume that the complete sequence $\{p_m^{(n)}\}_{m=1}^{\infty}$ is contained in R_{k_n}. On the other hand, if each R_k contains finitely many elements of the sequence $\{p_m^{(n)}\}_{m=1}^{\infty}$, we make $k_n = 0$.

Let $w_0 \in S$ be a limit point of the sequence $\{w_n\}_{n=1}^{\infty}$. By continuity of f, for each $n \in \mathbb{N}$, $f(w_n) = \{q_0\}$. Thus $f(w_0) = \{q_0\}$. In particular, $f(w_0) \neq S$. Thus $w_0 \in S - K$. Let G be the component of $S - K$ which contains the point w_0.

Let $\varphi : [0,1] \to S$ be a continuous map such that $\varphi|(0,1) : (0,1) \to G$ is a homeomorphism and $\varphi(0)$, $\varphi(1) \in K$. By the choice of w_0, we can choose three differente numbers n, r and s in $\mathbb{N}$ such that w_n, w_r and w_s belong to G. One of the two components of $G - \{w_n\}$ (or both) does not contain the point z_{k_n}. So one of the two intervals $(0, \varphi^{-1}(w_n))$ or $(\varphi^{-1}(w_n), 1)$ (or both) does not intersect the set $\varphi^{-1}(z_{k_n})$. The same happens with each one of the numbers r and s. Since we have three numbers and only two choices for the intervals, we may assume that the interval $(0, \varphi^{-1}(w_n))$ does not intersect the set $\varphi^{-1}(z_{k_n})$ and the interval $(0, \varphi^{-1}(w_r))$ does not intersect the set $\varphi^{-1}(z_{k_r})$. Let $b = \varphi^{-1}(w_n)$ and $c = \varphi^{-1}(w_r)\}$.

Let $\eta = \|z_n - z_r\| > 0$. Since $\varphi(0) \in K$, $f(\varphi(0)) = S$. Thus there exists $a \in (0, \min\{b, c\})$ such that $H(f(\varphi(a)), S) < \frac{\eta}{3}$.

Claim 2. $f(\varphi(a)) \subset S$ and $f(\varphi(a)) \in \mathrm{cl}_{C(X)}(C(R_n)) \cap \mathrm{cl}_{C(X)}(C(R_r))$.

Notice that there exists a retraction $\rho : X \to S$. Since $K = f^{-1}(\{S\})$ and G is a component of $S - K$, for each $t \in [a, b]$, $f(\varphi(t)) \neq S$. Since S is a terminal subcontinuum of X, by the proof of Theorem 11.5 of [3], it follows that $C(S) - \{S\}$ is an arcwise component of $C(X) - \{S\}$. Since $f(\varphi([a, b])) \subset C(X) - \{S\}$ and $f(\varphi(b)) = f(w_n) = \{q_0\} \in C(S) - \{S\}$, we conclude that $f(\varphi([a, b])) \subset C(S) - \{S\}$. In particular, $f(\varphi(a)) \in C(S)$Thus, for each $t \in [a, b]$, $\rho(f(\varphi(t))) = f(\varphi(t)) \in C(S) - \{S\}$. Let $\varepsilon_0 = \min\{H(\rho(f(\varphi(t))), S) : t \in [a, b]\} > 0$.

We only prove that $f(\varphi(a)) \in \mathrm{cl}_{C(X)}(C(R_n))$, the proof that $f(\varphi(a)) \in \mathrm{cl}_{C(X)}(C(R_r))$ is similar. Let $\varepsilon > 0$.

By the continuity of ρ and f, there exists $\lambda > 0$ such that, if $x, y \in X$ and $\|x - y\| < \lambda$, then $H(\rho(f(x)), \rho(f(y))) < \varepsilon_0$ and $H(f(x), f(y)) < \varepsilon$.

Apply Property (e) to the number λ and the path $\varphi|[a,b] : [a,b] \to S$, to obtain a number $\delta > 0$ with the mentioned properties.

By the choice of n, $z_{k_n} \notin \varphi([a,b])$. So $\varphi([a,b]) \subset S - \{z_{k_n}\}$. In the case that $k_n = 0$, there exists $N \in \mathbb{N}$ such that, for each $k \geq N$, $z_k \notin \varphi([a,b])$. In this case $R_1 \cup \ldots \cup R_N$ contains finitely many elements of the sequence $\{p_m^{(n)}\}_{m=1}^{\infty}$. Thus, we can choose $m \in \mathbb{N}$ such that $p_m^{(n)} \notin R_1 \cup \ldots \cup R_N$ and $\|w_n - p_m^{(n)}\| < \delta$. Hence, $p_m^{(n)} \in R_{N_0}$ for some $N_0 > N$, so $z_{N_0} \notin \varphi([a,b])$. In the case that $k_n \neq 0$, let $N_0 = k_n$ and choose $m \in \mathbb{N}$ such that $\|w_n - p_m^{(n)}\| < \delta$ and $p_m^{(n)} \in R_{k_n} = R_{N_0}$. In any case, numbers N_0 and m can be chosen in such a way that $\|w_n - p_m^{(n)}\| < \delta$, $p_m^{(n)} \in R_{N_0}$ and $z_{N_0} \notin \varphi([a,b])$.

By the choice of δ, there exists a path $\beta : [a,b] \to R_{N_0}$ such that $\beta(b) = p_m^{(n)}$ and $\|\varphi(t) - \beta(t)\| < \lambda$ for each $t \in [a,b]$. By the choice of λ, given $t \in [a,b]$, $H(\rho(f(\varphi(t))), \rho(f(\beta(t)))) < \varepsilon_0$ and $H(f(\varphi(t)), f(\beta(t))) < \varepsilon$. Thus, $H(f(\varphi(a)), f(\beta(a))) < \varepsilon$ and, by the choice of ε_0, $\rho(f(\beta(t))) \neq S$ for each $t \in [a,b]$.

By definition, $f(\beta(b)) = f(p_m^{(n)}) = \{q_m^{(n)}\} \subset R_n$. We need to prove that $f(\beta(a)) \subset R_n$. In order to prove this inclusion, we prove that $f(\beta(t)) \subset R_n$ for each $t \in [a,b]$. Suppose to the contrary that there exists $t \in [a,b]$ such that $f(\beta(t)) \cap (X - R_n) \neq \emptyset$. Since R_n is open in X, there exists $t_0 \in [a,b]$ such that $f(\beta(t_0)) \cap (X - R_n) \neq \emptyset$ and $f(\beta(t)) \subset R_n$ for each $t \in [t_0, b]$. Notice that $t_0 < b$. Since $S \cup R_n$ is compact, by continuity, $f(\beta(t_0)) \subset S \cup R_n$. Thus $f(\beta(t_0)) \cap S \neq \emptyset$. Given $t \in [t_0, b]$, consider the set $E_t = \bigcup\{f(\beta(u)) : u \in [t_0, t]\}$. Then $E_t \in C(X)$ (see [3, Lemmas 1.43 and 1.48]) and $\lim_{t \to t_0} E_t = E_{t_0} = f(\beta(t_0))$. Notice that, if $t > t_0$, then $f(\beta(t)) \subset E_t$, so $E_t \cap R_n \neq \emptyset$. Moreover, $f(\beta(t_0)) \subset E_t$, so $E_t \cap S \neq \emptyset$. By Property (h), $S \subset E_t$. Taking the limit as $t \to t_0$, we obtain that $S \subset E_{t_0} = f(\beta(t_0))$. Thus $S = \rho(f(\beta(t_0)))$, which is a contradiction with the conclusion in the previous paragraph. Therefore, $f(\beta(a)) \subset R_n$.

Hence, $f(\beta(a)) \subset R_n$ and $H(f(\varphi(a)), f(\beta(a))) < \varepsilon$. We have shown that $f(\varphi(a)) \in \mathrm{cl}_{C(X)}(C(R_n))$. This ends the proof of Claim 2.

We are ready to obtain the final contradiction. By the first part of Claim 2, $f(\varphi(a))$ is a subarc of S. By Claim 2 and Property (f) $z_n \notin f(\varphi(a))$ or z_n is an end point of $f(\varphi(a))$ and the same happens for z_r. Thus $f(\varphi(a))$ is contained in one of the two subarcs in which S is divided by the two points z_n and z_r. But this is impossible since $H(f(\varphi(a)), S) < \frac{\eta}{3}$. This completes the proof that there is no a continuous map from X onto $C(X)$.

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