The S_4 continua in sense of Michael are precisely the dendrites

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Abstract

We show that a continuum is an S_4 space in the sense of Micheal if and only if it is a dendrite.

1 Introduction

Michael [4, p.178] defined an S_4 space to be a space X such that there is a continuous selection $f: \mathcal{S} \to X$ for every every partition $\mathcal{S}$ of X into nonempty compact sets. The notion of a weak S_4 space, defined in [2], is the same as that for S_4 space except we assume the members of $\mathcal{S}$ have at most two points. The question of which spaces are S_4 spaces is asked in [4, p.155], and some partial answers are given in [4, pp.178–179]. In particular, it is mentioned that no simple closed curve is an S_4 space and that all finite trees are S_4 spaces. The particular question of whether S_4 continua are dendrites is due to Gail S. Young [7]. In [1], it is shown that every S_4 continuum is a dendrite. Our purpose here is to prove that dendrites actually characterize the S_4 continua.

Theorem 1 *The following conditions are equivalent for a continuum X:*

(a) *X is an S_4 space*

(b) *X is a weak S_4 space*

(c) *X is a dendrite.*

The implication (a)→(b) is immediate from the definitions. The implication (b)→(c) is shown in [1]. This paper will be dedicated to the implication (c)→(a). The implication will follow from from a general theorem on the existence of continuous selectors and a structure theorem on partitions of dendrites.

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2 Terminology

By ω we denote the non-negative integers.

For a set $A \subseteq X$ we write $\mathrm{cl}(A)$, $\mathrm{int}(A)$, $\mathrm{bd}(A)$ for the topological closure, interior, and boundary of A in X, respectively. If a space X can be write as the union of two disjoint nonempty closed sets A and B we say that A and B form a separation of X and write $X = A|B$. If $C \subseteq X$ and $X \setminus C$ is not connected we say that C is a separator of X. A maximal connected subset of a space X is called a component of X. Given a point $x \in X$ the quasicomponent of x is the intersection of all clopen subsets of X containing x. It is well known that every component of a space is contained a quasicomponent of the space.

By a compactum we mean a nonempty compact metric space. A continuum connected compactum. A space is a dendrite provided that is a locally connected continuum containing no simple closed curve. Every connected subset of a dendrite is arcwise connected [5, 9.10]. Given two points x and y in a dendrite X we let $[x, y]$. denote the unique arc in X with endpoints x and y. We say a space X is regular provided that every point of X has a local base of open neighborhoods with finite boundary.

Suppose X is a metric space with metric d. The diameter of a nonempty set $A \subseteq X$ is defined by $\mathrm{diam}(A) = \sup\{d(x, y) \colon x, y \in A\}$. Given nonempty sets $A, B \subseteq X$ we define $\underline{\mathrm{d}}(A, B) = \inf(\{d(x, y) \colon x \in A \;\&\; y \in B\})$. Given sets $A, B \subseteq X$ we define the Hausdorff distance between A and B to be

$$\mathrm{H}(A, B) = \max(\sup(\{\underline{\mathrm{d}}(\{x\}, B) \colon x \in A\}), \sup(\{\underline{\mathrm{d}}(A, \{y\}) \colon y \in B\})).$$

When H is restricted to the compact subsets of X it is a metric known as the Hausdorff metric. We denote the space of compacta with the Hausdorff metric (equivalently the Vietoris topology) by 2^X. Recall that a basic open set in the Vietoris topology has the form

$$< U_1, \ldots, U_n >= \{A \in 2^X \colon A \subseteq \bigcup_{i=1}^{n} U_i \text{ and } A \cap U_i \neq \emptyset \text{ for } 1 \leq i \leq n\}.$$

Where U_i is a nonempty open subset of X for each $1 \leq i \leq n$.

Given a space X and $\mathcal{S} \subseteq 2^X$ we say that $h \colon \mathcal{S} \to X$ is a selector provided that the cardinality of $h(S) \in S$ for all $S \in \mathcal{S}$. If h is continuous we say h is continuous selector.

3 Results

We say a family $\mathcal{F}$ of disjoint sets in space X is manageable provided that $\bigcup \mathcal{F}$ is closed in X and for every $F \in \mathcal{F}$ there is an open set U such that $F \subseteq U$ and $U \cap (\bigcup(\mathcal{F} \setminus \{F\})) = \emptyset$. A partition $\mathcal{S}$ of a metric space X into compacta (considered as a subset of 2^X) is said to be admissible provided that:

(C1) $\mathcal{S}$ is regular,

(C2) if $S \in \mathcal{S}$ has a nondegenerate component then $\{S\}$ is a component of $\mathcal{S}$,

(C3) any two distinct points of $\mathcal{S}$ can be separated by a third point in $\mathcal{S}$, and

(C4) for every $\epsilon > 0$ the collection of all components $\mathcal{Q}$ of $\mathcal{S}$ such that some component of $\bigcup \mathcal{Q}$ has diameter at least ϵ is manageable.

Theorem 2 *If X is a separable metric space and $\mathcal{S}$ is an admissible partition of X into compacta, then there is for every $p \in X$ a continuous selector $h\colon \mathcal{S} \to X$ such that $p \in h[\mathcal{S}]$.*

Theorem 3 *If X is a dendrite, then every partition of X into compacta is admissible.*

The implication (c)→(a) of Theorem 1 follows immediately from Theorem 2 and Theorem 3.

We note the following corollary of Theorem 2 which may be of interest.

Corollary 4 *Let X be a separable metric space and $\mathcal{C}$ be a partition of X into totally disconnected compacta that is homeomorphic to a connected subset of a dendrite, then there is for every $p \in X$ a continuous selector $h\colon \mathcal{C} \to X$ such that $p \in h[\mathcal{C}]$.*

If X and Y are compacta and $f\colon X \to Y$ is an open map, then $\{f^{-1}(y)\colon y \in f[X]\} \subseteq 2^X$ is a partition of X which is homeomorphic to $f[X]$. With this fact in mind, Corollary 4 can be seen as a generalization of the following result of Whyburn.

Proposition 5 *([8, Chap. 10, 2.4]) Let X be a compactum Y be a dendrite and $f\colon X \to Y$ be a continuous light open map. For every $p \in X$ there is a continuum $W \subseteq X$ such that $p \in W$, $f|W\colon W \to f[W]$ is a homeomorphism.*

4 Components and Quasicomponents of regular spaces

We say that X has property P provided that for any $x \in X$ if Q is the quasi-component of x and U is an open neighborhood of x, then there is an open set V such that $x \in V \subseteq U$ and $\mathrm{bd}(V) \subseteq Q$.

The spaces satisfying property P is a fairly large class.

Lemma 6 *Let X be a topological space. If every $x \in X$ has a neighborhood base consisting of open sets U such that $\mathrm{bd}(U)$ is contained the union of finitely many quasicomponents of X, then X has property P.*

Proof. Let $U \subseteq X$ and $x \in X$. Let Q be the quasicomponent of x. By assumption, there is an open neighborhood V_1 of x such that $\mathrm{bd}(V_1)$ is contained in the union of finitely many quasicomponents of X and $V_1 \subseteq U$. Let $C_1, \ldots, C_n$

be the quasicomponents of X such that $C_i \cap (\mathrm{bd}(V_1) \setminus Q) \neq \emptyset$. For each C_i there is a clopen set E_i such that $C_i \subseteq E_i$ and $Q \cap E_i = \emptyset$. Let $V = V_1 \setminus \bigcup_{i=1}^n E_i$. Clearly, V is an open neighborhood of x and $V \subseteq U$. Notice that $X \setminus V = (X \setminus V_1) \cup (\bigcup_{i=1}^n E_i)$. Also, $\mathrm{cl}(V) = \mathrm{cl}(V_1 \setminus (\bigcup_{i=1}^n E_i)) = \mathrm{cl}(V_1) \setminus (\bigcup_{i=1}^n E_i)$. Thus, $\mathrm{bd}(V) = \mathrm{cl}(V) \cap (X \setminus V) \subseteq \mathrm{bd}(V_1) \setminus (\bigcup_{i=1}^n E_i) \subseteq Q$. ■

Lemma 7 *Suppose X is Lindelof, and has property P. If $Q \subseteq X$ is a quasicomponent of X and $U \subseteq X$ is open and $Q \subseteq U$, then there is a clopen set V such that $Q \subseteq V \subseteq U$.*

Proof. Let Q be a quasicomponent of X. By property P we may find for each $x \in Q$ find an open set U_x in X such that $x \in U_x$, $\mathrm{bd}(U_x) \subseteq Q$, and $U_x \subseteq U$. Let $\mathcal{U} = \{U_x \colon x \in Q\} \cup \{U \colon U \text{ is clopen and } U \cap Q = \emptyset\}$. Since $\mathcal{U}$ is an open cover of X, there is a countable subcover $\{U_n\}_{n\in\omega}$ of X. We define a cover $\{V_n\}_{n\in\omega}$ as follows:

$$V_n = \begin{cases} U_n \setminus \bigcup_{i=0}^n U_i & \text{if } U_n \cap Q = \emptyset; \\ U_n \setminus (\bigcup\{U_i \colon i < n \text{ and } U_i \cap Q = \emptyset\}) & \text{if } U_n \cap Q \neq \emptyset. \end{cases}$$

Notice that if $V_k \cap V_n \neq \emptyset$, then either $n = k$ or both V_k and V_n have nonempty intersection with Q.

For n such that $U_n \cap Q \neq \emptyset$ we have $V_n \cap (X \setminus Q)$ clopen in $X \setminus Q$. With this observation it is easy to show that V_n is open for every $n \in \omega$. The observation also can be used to show that if $U_n \cap Q = \emptyset$, then V_n is clopen in X. Also, if $U_n \cap Q = \emptyset$, then $V_n \cap (\bigcup_{i\neq n} V_i) = \emptyset$.

Let $V = \bigcup\{V_n \colon V_n \cap Q \neq \emptyset\}$. Clearly, V is open and $Q \subseteq V \subseteq U$. We will be done if we show that V is closed. Suppose $y \in \mathrm{cl}(V)$. Let $k \in \omega$ be such that $y \in V_k$. Since $y \in \mathrm{cl}(V)$, there is an n such that $V_n \cap Q \neq \emptyset$ and $V_n \cap V_k \neq \emptyset$. By the statement immediately following the definition of $\{V_n\}_{n\in\omega}$, we have that $V_k \cap Q \neq \emptyset$. So, $V_k \subseteq V$ by definition of V. Thus, $y \in V$ showing that V is closed. ■

Lemma 8 *If X is Lindelof, normal, and has property P, then the quasicomponents and components of X are the same.*

Proof. Let Q be a quasicomponent of X. By way of contradiction, assume that Q is not connected. Since Q is closed and X is normal, $Q = (U \cap Q)|(V \cap Q)$ where U and V are disjoint open sets in X. By Lemma 7, there is a clopen set W such that $Q \subseteq W \subseteq U \cup V$. Since $U \cap V = \emptyset$, $U \cap W$ is clopen. Now $U \cap Q \subseteq W \cap U$ and $V \cap Q \subseteq X \setminus (W \cap U)$ contradicting that Q is a quasicomponent. ■

For the proof of Theorem 2 we note the following corollary.

Corollary 9 *Let X be a separable metric space and $\mathcal{S}$ be an admissible partition of X into compacta. The quasicomponents and components of any subset of $\mathcal{S}$ are the same. Also, for every component $\mathcal{Q}$ of $\mathcal{S}$ and open set $\mathcal{U}$ containing $\mathcal{Q}$ there is a clopen set $\mathcal{V}$ such that $Q \subseteq \mathcal{V} \subseteq \mathcal{V}$.*

Proof. By (C1) any subset $\mathcal{T}$ of $\mathcal{S}$ is a regular separable metric space. The corollary now follows by Lemma 6, Lemma 8, and Lemma 7. ■

5 Proof of Theorem 3

Let X be a space and $\mathcal{S}$ be a partition of X into compacta. The membership function $F\colon X \to \mathcal{S}$ is the function defined by $x \in F(x)$. The next lemma shows that the membership function has a property close to confluence.

Lemma 10 *Suppose X is a topological space and $\mathcal{S}$ is a partition of X into compacta with membership function F. If $\mathcal{Q} \subseteq \mathcal{S}$ is connected and C is a quasicomponent of $\bigcup \mathcal{Q}$, then $F(C) = \mathcal{Q}$.*

Proof. By way of contradiction, assume there is a $M \in \mathcal{Q}$ such that $M \notin F(C)$. Since $C \cap M = \emptyset$, there is for each $m \in M$ a separation $U^m|V^m$ of $\bigcup \mathcal{Q}$ such that $C \subseteq U^m$ and $m \in V^m$. Since M is compact, there are finitely many $m_1, \ldots, m_k \in M$ such that $M \subseteq \bigcup_{i=1}^k V^{m_i}$. Let $V = \bigcup_{i=1}^k V^{m_i}$ and $U = \bigcap_{i=1}^k U^{m_i}$. Notice that $\bigcup \mathcal{Q} = U|V$ and $C \subseteq U$ and $M \subseteq V$. Notice that $F(C) \subseteq < U, X >$ and $M \notin < U, X >$. By connectedness, $\mathcal{Q} \cap \mathrm{bd}(< U, X >) \neq \emptyset$. Let $E \in \mathcal{Q} \cap \mathrm{bd}(< U, X >)$. Notice that $E \cap \mathrm{bd}(U) \neq \emptyset$, contradicting that $U|V$ is a separation of $\bigcup \mathcal{Q}$. ■

Lemma 11 *If X is regular and $\mathcal{S}$ is a partition of X into compacta, then $\mathcal{S}$ is regular.*

Proof. Let $S \in \mathcal{S}$. Since X is regular and S is compact there is a local base for S with open sets of the form $\mathcal{U} =< U_1, \ldots U_n >$ where each U_i has finite boundary. Notice that any $T \in \mathrm{bd}(\mathcal{U})$ must have nonempty intersection with $\bigcup_{i=1}^n \mathrm{bd}(U_i)$. Since $\bigcup_{i=1}^n \mathrm{bd}(U_i)$ is finite and $\mathcal{S}$ is a partition, $\mathrm{bd}(\mathcal{U})$ is finite. So, $\mathcal{S}$ is regular. ■

Lemma 12 *Let X be a dendrite and $S, T \subseteq X$ be disjoint compacta. Either there is a $s \in S$ such that T is contained in a component of $X \setminus \{s\}$ or there is a $t \in T$ such that S is contained in a component of $X \setminus \{t\}$.*

Proof. Assume that there is no $s \in S$ or $t \in T$ with the desired property. Let $t_0 \in T$ and $s_0 \in S$. Let A_0 be the arc $[t_0, s_0]$. By our assumption about S, there is a component C_0 of $X \setminus \{s_0\}$ such that $[t_0, s_0] \cap C_0 = \emptyset$ and $C_0 \cap T \neq \emptyset$. Let A_1 be the arc $[s_0, t_1]$ where $t_1 \in C_0$. Since $(s_0, t_1] \subseteq C_0$, $A_0 \cup A_1 = [t_0, t_1]$. By assumption, S is not contained in any component of $X \setminus \{t_1\}$. Let s_1 be an element of S that is in a component C_1 of $X \setminus \{t_1\}$ that does not contain $[t_0, t_1)$. Let A_2 be the arc $[t_1, s_1]$. Notice that $A_0 \cup A_1 \cup A_2 = [t_0, s_1]$ since $C_1 \cap (A_0 \cup A_1) = \emptyset$ and $(t_1, s_1] \subseteq C_1$. Moreover, $A_0 \cap A_2 = \emptyset$.

By our assumptions on S and T we may continue this process indefinitely to get an infinite sequence of mutually disjoint arcs $\{A_{2n}\}_{n \in \omega}$ of the form $[t_n, s_n]$.

Since X is a dendrite, the diameters of these arcs must tend to zero. So, by compactness of S and T, we have $S \cap T \neq \emptyset$ a contradiction. ■

Lemma 13 *Let X be a dendrite and $\mathcal{S}$ be a partition of X into disjoint compacta. If $S, T \in \mathcal{S}$ are distinct points, then there is a $R \in \mathcal{S}$ that separates S from T.*

PROOF. By Lemma 12, we may assume that there is a $t \in T$ such that S is contained in a component D of $X \setminus \{t\}$. Since t is not a cutpoint of $D \cap \{t\}$, t is an endpoint of $D \cup \{t\}$ by [5, 10.7]. So, there is a point $p \in D$ such that p separates t from S in $D \cup \{t\}$. By unicoherence, p separates t from S in X. Let $R = F(p)$. By way of contradiction assume that S and T are in the same component $\mathcal{C}$ of $\mathcal{S} \setminus \{R\}$. Let E be the component of $\bigcup \mathcal{C}$ which contains t. By Lemma 10, $S \in F[E]$. On the other hand, E is contained in the component G of $X \setminus \{p\}$ which contains t. Since $G \cap S = \emptyset$, $S \notin F[E]$, a contradiction. Since S and T are in different components of $\mathcal{S} \setminus \{R\}$, S and T are in different quasicomponents of $\mathcal{S} \setminus \{R\}$, by Corollary 9 and Lemma 11. Thus, R separates S and T. ■

Lemma 14 *Let X be a metric space and $\mathcal{S}$ be a partition of X into compacta with membership function F and $\mathcal{C} \subseteq \mathcal{S}$ be connected. If $S \in \mathcal{C}$ and there is an $s \in S$ and an open $U \subseteq X$ such that $s \in U$ and $\mathrm{bd}(U) \subseteq S$, then $F[U] = \mathcal{S}$.*

PROOF. Clearly, $S \in < U, X >$. Suppose $R \notin < U, X >$ and $R \in \mathcal{C}$. Since $\mathcal{C}$ is connected, there is a $T \in \mathrm{bd}(< U, X >) \cap \mathcal{C}$. So, $T \cap \mathrm{bd}(U) \neq \emptyset$. Since $\mathrm{bd}(U) \subseteq S$ and $\mathcal{S}$ is a partition, $T = S$. So, $S \in \mathrm{bd}(< U, X >)$ and $S \in < U, X >$ a contradiction. ■

PROOF OF THEOREM 3 Let X be a dendrite, $\mathcal{S}$ be a partition of X into compacta, and F be the membership function for the partition.

By Lemma 11 $\mathcal{S}$ is regular. So, we have (C1).

Suppose $S \in \mathcal{S}$ contains a nondegenerate component M. Let $\mathcal{C}$ be the component of S in $\mathcal{S}$. Since X is a dendrite there is point $p \in M$ such that p is a cutpoint of both X and M, and p has a base of open neighborhoods U such that $\mathrm{bd}(U)$ has exactly two points [5, 10.42]. Let $x, w \in M$ be separated by p and U an open neighborhood of p such that $\mathrm{bd}(U)$ has exactly two points and $w, x \notin U$. Since X is unicoherent and p separates x and w, we have $[x, p] \cap [p, w] = \{p\}$. It is now easy to see that $\mathrm{bd}(U) \subseteq M \subseteq S$. By Lemma 14, $F[U] = \mathcal{C}$. Since we may choose U to be as small we like and the elements of $\mathcal{C}$ are compact, we see that $p \in \bigcap \mathcal{C}$. Thus, $\mathcal{C} = \{S\}$. So, we have (C2).

By Lemma 13, any two points of $\mathcal{S}$ are separated by a third point. So, we have (C3)

Let $\epsilon > 0$ and θ be the collection of components $\mathcal{C}$ of $\mathcal{S}$ such that some component $D_{\mathcal{C}}$ of $\bigcup \mathcal{C}$ has diameter at least ϵ. Since X is a dendrite and $\mathcal{S}$ is a partition, the collection $\{D_{\mathcal{C}} \colon \mathcal{C} \in \theta\}$ is finite [3]. Since $\mathcal{S}$ is a partition and θ is a mutually disjoint collection, θ is finite. Since components are closed, θ is manageable. So, we have (C4).

Therefore, $\mathcal{S}$ is admissible. ■

6 Proof of Theorem 2

Lemma 15 *If a metric space Y is regular and connected, then Y is locally connected.*

Proof. If Y is just a single point, then Y is obviously locally connected. So, we assume that Y contains at least two points. Let $y \in Y$ and $x \in Y \setminus \{y\}$. Let V be open neighborhood of y. Let $U \subseteq V \setminus \{x\}$ be an open neighborhood of y with finite boundary such that $\mathrm{cl}(U) \subseteq V$. Let R be a quasicomponent of $\mathrm{cl}(U)$.

Suppose that $R \cap \mathrm{bd}(U) = \emptyset$. Since $\mathrm{bd}(U)$ is finite, there would be a separation $Z|W$ of $\mathrm{cl}(U)$ such that $\mathrm{bd}(U) \subseteq Z$ and $R \subseteq W$. Notice that W is clopen and nonempty in Y and $x \notin W$, contradicting that Y is connected. Thus, if R is a quasicomponent of $\mathrm{cl}(U)$, then $R \cap \mathrm{bd}(U) \neq \emptyset$.

Since $\mathrm{bd}(U)$ is finite, there are only finitely many quasicomponents R of $\mathrm{cl}(U)$. Thus, each quasicomponent of $\mathrm{cl}(U)$ is a component of $\mathrm{cl}(U)$. Let T be the component of y in $\mathrm{cl}(U)$. Since $y \in U$ and T is open in $\mathrm{cl}(U)$, T is a connected neighborhood of y contained in V. ■

A compact space W is called a perfect compactification of a space X provided that X is dense in W and for any closed subset C of X if C separates two subsets A and B of X, then $\mathrm{cl}_W(C)$ separates A and B in W. A space X is said to be rim compact provided that every point of X has a base of open sets with compact boundaries. In particular, regular spaces are rim compact.

Proposition 16 *([6, Thm. 4.2, Cor 4.5]) Let X be a separable metric space. If X is connected, locally connected, and the components and quasicomponents of each subspace of X are the same and X is rim compact, then there is a hereditarily locally connected continuum W which is a perfect compactification of X and $W \setminus X$ contains no nondegenerate continuum.*

Lemma 17 *Let $\mathcal{S}$ be an admissible partition of a separable metric space X into disjoint compacta. If $\mathcal{C} \subseteq \mathcal{S}$ is connected, then $\mathcal{C}$ is homeomorphic to a connected subset of a dendrite.*

Proof. By (C1) and Lemma 15, $\mathcal{C}$ is locally connected and rim compact. By (C1) and Corollary 9, the components and quasicomponents of each subspace of X are the same. By Proposition 16, there is a hereditarily locally connected continuum W which is a perfect compactification of $\mathcal{C}$ and $W \setminus \mathcal{C}$ contains no nondegenerate continuum. Suppose W contains a simple closed curve M. Since $W \setminus \mathcal{C}$ contains no nondegenerate continuum, there exist distinct $S, T \in M \cap \mathcal{C}$. By Lemma 13, there is a point $R \in \mathcal{C}$ such that R separates S and T in $\mathcal{C}$. Since W is a perfect compactification of $\mathcal{C}$, R also separates S and T in W, contradicting that S and T lie on M. Since W contains no simple closed curve, we conclude that W is a dendrite. ■

Lemma 18 *Let X be a metric space and $\mathcal{S}$ be a partition of X into compacta and F be the membership function. If $\mathcal{C} \subseteq \mathcal{S}$ is a nondegenerate continuum, then $F|\bigcup\mathcal{C}\colon \bigcup\mathcal{C} \to \mathcal{C}$ is continuous and open.*

Proof. Let $x \in \bigcup\mathcal{C}$ and $\{x_n\}_{n\in\omega}$ be a sequence on $\bigcup\mathcal{C}$ such that $\lim x_n = x$. Since $\mathcal{C}$ is compact, there is a $P \in \mathcal{C}$ such that some subsequence of $\{F(x_n)\}_{n\in\omega}$ converges to P. Since $x_n \in F(x_n)$ for every $n \in \omega$, it follows that $x \in P$. Since $\mathcal{S}$ is a partition, $P = F(x)$. So, some subsequence of $\{F(x_n)\}_{n\in\omega}$ converges to $F(x)$. So, $F|\bigcup\mathcal{C}$ is continuous.

Let $x \in X$ and U be an open neighborhood of x. It is immediate from the definition of F that $F[U] =< U, X > \cap\mathcal{S}$ which is open. So, F is open. Since $F|\bigcup\mathcal{C} = F|F^{-1}(\mathcal{C})$, $F|\bigcup\mathcal{C}\colon \bigcup\mathcal{C} \to \mathcal{C}$ is open. ■

Lemma 19 *Let X be a metric space and $\mathcal{S}$ be a partition of X into disjoint compacta satisfying (C2) with membership function F. Suppose $\mathcal{C} \subseteq \mathcal{S}$ is homeomorphic to a nondegenerate connected subset of a dendrite. If $\{S_n\}_{n\in\omega}$ is sequence on $\mathcal{C}$ and $\lim S_n = S \in \mathcal{C}$, then*

$$\lim_{n\in\omega} \sup \left\{ \mathrm{diam}(D)\colon D \textit{ is a component of } \bigcup[S_n, S] \right\} = 0.$$

Proof. Since $\mathcal{C}$ is contained in a dendrite, $\lim[S_n, S] = \{S\}$. Since F is open, $\lim F^{-1}([S_n, S]) = F^{-1}(\{S\}) = S$. By (C2), S is totally disconnected. Since S is totally disconnected and $F^{-1}([S_n, S])$ is compact for every n, it follows that $\lim_{n\in\omega} \sup \{\mathrm{diam}(D)\colon D$ is a component of $\bigcup[S_n, S]\} = 0$. ■

Lemma 20 *If X is a separable metric space and $\mathcal{C}$ is a connected subset of an admissible partition of X into disjoint compacta, then for every $a \in \bigcup\mathcal{C}$ there is a continuous selector h for $\mathcal{C}$ such that $a \in h[\mathcal{C}]$.*

Proof. If $\mathcal{C}$ is a single point then the lemma is obviously true. So, we assume throughout that $\mathcal{C}$ is nondegenerate. In particular, condition (C2) in the definition of admissablity implies that the restricted membership function $F\colon \bigcup\mathcal{C} \to \mathcal{C}$ has totally disconnected point-inverses.

Let θ be the collection of all nonempty $M \subseteq \bigcup\mathcal{C}$ such that $a \in M$, $F|M$ is one-to-one and for any two points $p, q \in M$ there is an arc $A_{p,q} \subseteq M$ from p to q such that $F|A_{p,q}$ is continuous. It is easily checked that θ satisfies the hypothesis of the Hausdorff Maximal Principle. Let M be a maximal element of θ.

We claim that $F[M]$ is closed in $\mathcal{C}$. By way of contradiction, assume that $S \in \mathrm{cl}(F[M]) \setminus F[M]$. Since $F[M]$ is connected, $F[M] \cup \{S\}$ is connected. By Lemma 17, $\mathcal{C}$ can be embedded into a dendrite. So, $F[M] \cup \{S\}$ is arcwise connected. Let $s_0 \in M$. The arc $[F(s_0), S]$ is contained in $F[M] \cup \{S\}$. Let $\{S_n\}_{n\in\omega}$ be a sequence of points on $[F(s_0), S]$ such that $S_0 = F(s_0)$, $\lim S_n = S$ and $[F(s_0), S_n] \subseteq [F(s_0), S_{n+1})$. Let $s_n \in S_n \cap M$ for every n. Taking a subsequence if necessary we may assume that there is an $s \in S$ such that $\lim s_n = s$. Let $n \in \omega$. Since $M \in \theta$, $F|A_{s_n,s_{n+1}}$ is a homeomorphism to the arc $[S_n, S_{n+1}]$. So, $A_{s_n,s_{n+1}} \subseteq F^{-1}([S_n, S_{n+1}]) \subseteq F^{-1}([S_n, S])$. Since $F|M$ is one-to-one, $\bigcup_{n\in\omega} A_{s_n,s_{n+1}}$ is homeomorphic to the halfline $[S_0, S)$. Since $A_{s_n,s_{n+1}} \subseteq F^{-1}([S_n, S])$, Lemma 19 implies that $\lim \operatorname{diam}(A_{s_n,s_{n+1}}) = 0$. Since $\lim s_n = s$, $A_{s_0,s} = \{s\} \cup (\bigcup_{n\in\omega} A_{s_n,s_{n+1}})$ is an arc and $F|A_{s_0,s} \colon A_{s_0,s} \to [S_0, S]$ is continuous. So, $s \cup M \in \theta$, a contradiction to maximality. Thus, $F[M]$ is closed.

We claim $F[M] = \mathcal{C}$. By way of contradiction, assume there is a $S \in \mathcal{C} \setminus f[M]$. Since $\mathcal{C}$ is arcwise connected and $F[M]$ is closed, there is a nondegenerate arc $[T, S] \subseteq \mathcal{C}$ such that $[T, S] \cap F[M] = \{T\}$. By Lemma 18 and (C2), $F|\bigcup[T, S]$ is continuous, open, and light. As a union of a compact collection of sets, $\bigcup[T, S]$ is compact. Let $t \in M \cap T$. By Proposition 5, there is an arc $A \subseteq \bigcup[T, S]$ such that $t \in A$ and $F|A$ is a homeomorphism. It is easy to check that $M \cup A \in \theta$, a contradiction to maximality.

Since $(F|M)^{-1} \colon \mathcal{C} \to M$ is a selector and $a \in M$, it remains show that $(F|M)^{-1}$ is continuous.

Let $[S, T]$ be an arc in $\mathcal{C}$. Let $s \in S \cap M$ and $t \in T \cap M$ and $A \subseteq M$ be an arc from s to t such that $f|A$ is continuous. Since $F|A$ is continuous and one-to-one and $\mathcal{C}$ contains no simple closed curve, $F[A] = [S, T]$. Thus, $(F|M)^{-1}|[S, T]$ is continuous. So, $(F|M)^{-1}$ is continuous on arcs.

Let $S \in \mathcal{C}$ and $\{S_n\}_{n\in\omega}$ be a sequence of distinct points on $\mathcal{C}$ such that $\lim S_n = S$. Since $\mathcal{C}$ is contained in a dendrite, $\lim[S_n, S] = \{S\}$. By Lemma 19, $\lim \operatorname{diam}(A_{s_n,s}) = 0$. Thus, $\lim(F|M)^{-1}(S_n) = \lim s_n = s = (F|M)^{-1}(S)$ showing that $(F|M)^{-1}$ is continuous. ∎

For the remainder of this paper we will assume that X is a fixed separable metric space with a fixed admissible partition $\mathcal{S}$ into compacta with membership function F. We will also fix $p \in X$. Moreover, we will assume (remetrizing if necessary) that the metric d on X has the property that $\operatorname{diam}(X) < 1$. In particular, $\operatorname{diam}(\mathcal{S}) < 1$.

For each $n \in \omega$ let J_n denote the collection of all components $\mathcal{Q}$ of $\mathcal{S}$ such that some component of $\bigcup \mathcal{Q}$ has diameter at least $1/2^n$. Notice that $J_0 = \emptyset$.

Lemma 21 *Every nondegenerate component of $\mathcal{S}$ is contained in $\bigcup_{n\in\omega} J_n$. The collection $\bigcup_{n\in\omega} J_n$ is countable.*

Proof. Suppose $\mathcal{Q}$ is a nondegenerate component of $\mathcal{S}$. By Lemma 10 and Corollary 9, there is an $n \in \omega$ such that $\mathcal{Q} \in J_n$. Since $\mathcal{S}$ is separable and J_n is manageable, J_k is countable for all $k \in \omega$. Obviously $\bigcup_{k\in\omega} J_k$ is countable. ∎

Lemma 22 *If a component $\mathcal{C}$ of $\mathcal{S}$ has diameter greater than ϵ, then some component of $\bigcup \mathcal{C}$ has diameter at greater than ϵ.*

Proof. Suppose $\operatorname{diam}(C) \le \epsilon$ for every component C of $\bigcup \mathcal{C}$. Let $S, T \in \mathcal{C}$. Let $s \in S$. Let C be the component of s in $\bigcup \mathcal{C}$. By Lemma 10 and Corollary 9, there is a $t \in T$ such that $t \in C$. So, $\underline{\mathrm{d}}(s, T) \le \epsilon$. A similar argument shows that $\underline{\mathrm{d}}(t, S) \le \epsilon$ for every $t \in T$. So, $\mathrm{H}(S, T) \le \epsilon$. Thus, $\operatorname{diam}(\mathcal{C}) \le \epsilon$. ∎

We denote the collection of all finite strings of non-negative integers (including the empty string $\emptyset$) by $\omega^{<\omega}$. By ω^n ($\omega^{\leq n}$) we denote the set of all elements of $\omega^{<\omega}$ of length exactly (less or equal) n. If $\sigma \in \omega^{<\omega}$ we let $|\sigma|$ denote the length (equivalently, cardinality) of σ. Given $\sigma, \rho \in \omega^{<\omega}$, we say that σ is the predecessor of ρ (or that ρ is the successor of σ) provided that $\sigma \subseteq \rho$ and $|\rho| = |\sigma| + 1$. Given $\sigma \in \omega^n$ and $i \in \omega$ we define $(\sigma * i) \in \omega^{n+1}$ so that $(\sigma * i)|n = \sigma$ and $(\sigma * i)(n) = i$.

We say a nonempty subset T of $\omega^{<\omega}$ is a tree provided that $\emptyset \in T$, for every $\sigma \in T$ there is at least one successor of σ in T, and every element of $T \setminus \{\emptyset\}$ has a predecessor. For $n \in \omega$ we denote $T \cap \omega^n$ by T_n. Given a tree T we let $T^\dagger \subseteq \omega^\omega$ denote the maximal $\subseteq$-chains in T.

Lemma 23 *There is a tree $T \subseteq \omega^{<\omega}$ and collections $\{\mathcal{V}_\sigma \colon \sigma \in T\}$ of clopen sets and $\{\mathcal{Q}_\sigma \colon \sigma \in T\}$ of components of $\mathcal{S}$ such that for every $n \in \mathbb{N}$ and $\tau, \sigma \in T$ we have:*

- *(A0)* $p \in \bigcup \mathcal{Q}_\emptyset$,
- *(A1)* $\tau \subseteq \sigma$ *implies* $\mathcal{V}_\sigma \subseteq \mathcal{V}_\tau$,
- *(A2)* $\mathcal{Q}_\sigma \subseteq \mathcal{V}_\sigma$,
- *(A3)* $\mathcal{V}_\sigma \cap \mathcal{V}_\tau \neq \emptyset$ *implies* $\tau = \sigma$,
- *(A4)* $\bigcup_{\xi \in T_n} \mathcal{V}_\xi = \mathcal{S}$,
- *(A5)* $\mathrm{H}(\mathcal{V}_\sigma, Q_\sigma) < 1/2^{|\sigma|}$, *and*
- *(A6)* $J_n \subseteq \{Q_\sigma \colon \sigma \in T_n\}$, *and*
- *(A7)* $\mathcal{Q}_{\sigma * 0} = \mathcal{Q}_\sigma$.

PROOF. Let $\mathcal{Q}_\emptyset$ be the component of $F(p)$ in $\mathcal{S}$. Let $\mathcal{V}_\emptyset = \mathcal{S}$. Keeping in mind that $\operatorname{diam}(\mathcal{S}) < 1$ we see that $T_0 = \{\emptyset\}$, $\mathcal{Q}_\emptyset$, and $\mathcal{V}_\emptyset$ satisfy (A0)-(A7)

Assume that $n \geq 1$ and we have defined $T_m \in \omega^m$ for every $0 \leq m \leq n-1$ so that $\bigcup_{m=0}^{n-1} T_m$, $\{V_\sigma \colon \sigma \in \bigcup_{m=0}^{n-1} T_m\}$ and $\{S_\sigma \colon \sigma \in \bigcup_{m=0}^{n-1} T_m\}$ satisfy conditions (A0)-(A7). We show how to define $T_n \subseteq \omega^{\leq n}$ so that (A0)-(A7) will be satisfied by $\bigcup_{m=0}^{n} T_m$.

Fix $\tau \in T_{n-1}$. Since $\mathcal{S}$ is admissible, J_{n+3} is mangable. Notice that $J_{n+3} \cup \{\mathcal{Q}_\tau\}$ is also mangable. Since $\mathcal{S}$ is separable and metric, $J_n \cup \{\mathcal{Q}_\tau\}$ is countable. Let Let $A \subseteq \omega$ and $\{\mathcal{P}_j \colon j \in A\}$ be an enumeration of the elements of J_{n+3} which are contained in $\mathcal{V}_\tau$ together with $\mathcal{Q}_\tau$. In the enumeration we will assume that $\mathcal{P}_0 = \mathcal{Q}_\tau$. Since $\{\mathcal{P}_j \colon j \in A\}$ is countable and manageable, we may use Corollary 9 and induction to construct a family $\{\mathcal{U}_j\}_{j \in A}$ of mutually disjoint clopen subsets of $\mathcal{V}_\tau$ such that $\mathcal{P}_j \subseteq \mathcal{U}_j$ and $\mathrm{H}(\mathcal{U}_j, Q_j) < 1/2^{2+j}$ for all j. It is easily checked that $\bigcup_{j \in A} \mathcal{U}_j$ is clopen in $\mathcal{S}$.

By Lemma 21 there is a $B \subseteq \omega$ such that the nondegenerate components of $\mathcal{V}_\tau \setminus (\bigcup_{j \in A} \mathcal{U}_j)$ may be enumerated as $\{\mathcal{C}_k \colon k \in B\}$. Notice that $\mathcal{C}_k \notin J_{n+3}$ for all $k \in B$.

Fix $k \in B$. By Corollary 9, there is a clopen neighborhood $\mathcal{T}^k$ of $\mathcal{C}_k$ such that $\mathcal{T}^k$ is contained in $\mathcal{S} \setminus \bigcup_{j\in A} \mathcal{U}_j$ and $\mathrm{H}(\mathcal{T}^k, \mathcal{C}_k) < 1/2^{n+3+k}$.

For each $k \in B$ let $\mathcal{R}^k = \mathcal{T}^k \setminus (\bigcup_{i<k} \mathcal{T}^k)$. Clearly, $\{\mathcal{R}^k \colon k \in \omega\}$ is a collection of disjoint clopen sets covering $\bigcup_{k\in\omega} \mathcal{C}_k$. Let $B_1 \subseteq B$ denote the set of all $k \in B$ such that $R^k \neq \emptyset$. Notice that $\mathcal{C}_k \subseteq \mathcal{R}^k \subseteq \mathcal{T}^k$ every $k \in B_1$. In particular, we have $\mathrm{H}(\mathcal{R}^k, \mathcal{C}_k) < 1/2^{n+3+k}$ for all $k \in B_1$.

Suppose now that $S \notin (\bigcup_{k\in B_1} \mathcal{R}^k) \cup (\bigcup_{j\in A} \mathcal{U}_j)$. Since $\{S\}$ is a component of $\mathcal{S}$, there is, by Corollary 9, a base of clopen neighborhoods of S. Pick a clopen neighborhood $\mathcal{K}_S$ of S such that $\mathrm{diam}(\mathcal{K}_S) < 1/2^{n+3}$ and $\mathcal{K}_S \cap (\bigcup_{j\in A} \mathcal{U}_j) = \emptyset$. Let $\mathcal{J}_S = \mathcal{K}_S \cup \{\mathcal{R}^k \colon \mathcal{R}^k \cap \mathcal{K}_S \neq \emptyset\}$. Notice that $\mathcal{J}_S \subseteq \mathcal{S} \setminus \bigcup_{j\in A} \mathcal{U}_j$.

We claim that $\mathrm{diam}(\mathcal{J}_S) < 1/2^n$. Let $P \in \mathcal{J}_S$. If $P \in \mathcal{K}_S$, then $\mathrm{H}(S, P) < 1/2^{n+3}$. Suppose $P \notin \mathcal{K}_S$. There is a $k \in B_1$ such that $P \in \mathcal{R}^k$. Since $\mathcal{R}^k \cap (\bigcup_{j\in A} \mathcal{U}_j) = \emptyset$, $\mathcal{C}_k \notin J_{n+3}$. By Lemma 22, $\mathrm{diam}(\mathcal{C}_k) \leq 1/2^{n+3}$. Thus, $\mathrm{diam}(\mathcal{R}^k) < 1/2^{n+2+k} + 1/2^{n+3} \leq 3/2^{n+3}$. So, $\mathrm{H}(P, S) < 1/2^{n+3} + 3/2^{n+3} = 1/2^{n+1}$. So, $\mathrm{diam}(\mathcal{J}_S) < 1/2^n$.

We claim that $\mathcal{J}_S$ is clopen in $\mathcal{S}$. Clearly, $\mathcal{J}_S$ is open. By way of contradiction, assume that $\mathcal{J}_S$ is not closed. Let $P \in \mathrm{cl}(\mathcal{J}_S) \setminus \mathcal{J}_S$. We may assume that there exists an increasing sequence $\{k_i\}_{i\in\omega}$ on B_1 and points $P_{k_i} \in \mathcal{R}^{k_i}$ such that $\lim_{i\in\omega} P_{k_i} = P$. Since $P \notin \mathcal{K}_S$, there is an $\epsilon > 0$ such that $\underline{\mathrm{H}}(\{P\}, \mathcal{K}_S) > \epsilon$. So, $\mathrm{diam}(\mathcal{R}^{k_i}) > \epsilon$ for almost all i. It follows that $\mathrm{diam}(\mathcal{C}_{k_i}) > \epsilon$ for almost all i. By Lemma 22, $\mathcal{C}_{k_i}$ has a component of diameter greater than ϵ for almost all i. Thus, $\mathcal{C}_{k_i} \subseteq J_l$ for some $l \in \omega$. Since J_l is manageable and $\lim_{i\in\omega} \underline{\mathrm{H}}(\mathcal{R}^{k_i}, \mathcal{C}_{k_i}) = 0$, $P \in \mathcal{C}_{k_i} \subseteq \mathcal{R}^{k_i}$ for some i. So, $P \in \mathcal{J}_S$, contradicting our assumption.

Let $\theta_1 = \{\mathcal{J}_S \colon S \notin (\bigcup_{k\in B_1} \mathcal{R}^k) \cup (\bigcup_{j\in A} \mathcal{U}_j)\}$. We may do a standard induction to find a $D \subseteq \omega$ and a refinement $\{\mathcal{L}_l \colon l \in D\}$ of θ_1 made up of mutually disjoint nonempty clopen sets such that $\bigcup\{\mathcal{L}_l \colon l \in D\} = \bigcup \theta_1$.

Define

$$\theta_2 = \{\mathcal{L}_l \colon l \in D\} \cup \left\{\mathcal{R}^k \colon \mathcal{R}^k \cap \left(\bigcup_{l\in D} \mathcal{J}_l\right) = \emptyset \text{ and } k \in B_1\right\} \cup \{\mathcal{U}_j \colon j \in A\}.$$

Notice θ_2 is a cover of $\mathcal{S}$ by mutually nonempty disjoint clopen sets. For every $j \in A$ let $\mathcal{V}_{\tau*3j} = \mathcal{U}_j$ and $\mathcal{Q}_{3j} = \mathcal{P}_j$. If $\mathcal{R}^k \in \theta_2$, then define $\mathcal{V}_{3k+1} = \mathcal{R}^k$ and $Q_{\tau*(3k+1)} = \mathcal{C}_k$. If $l \in D$, then define $\mathcal{V}_{\tau*(3l+2)} = \mathcal{L}_l$ and $Q_{\tau*(3l+2)}$ to be a component of $\mathcal{L}_l$. Let $T_\tau = \{\tau * (3k+1) \colon \mathcal{R}^k \in \theta_2\} \cup \{\tau * (3l+2) \colon l \in D\} \cup \{\tau * 3j \colon j \in A\}$.

For each $\tau \in T_{n-1}$ we perform a similar construction. Let $T_n = \bigcup_{\tau\in T_{n-1}} T_\tau$. Now $\bigcup_{m=0}^n T_m$, $\{\mathcal{V}_\sigma \colon \sigma \in \bigcup_{m=0}^n T_m\}$, and $\{\mathcal{Q}_\sigma \colon \sigma \in \bigcup_{m=0}^n T_m\}$ satisfy (A0)-(A7).

Finally, $T = \bigcup_{n\in\omega} T_n$, $\{\mathcal{V}_\sigma \colon \sigma \in T\}$, and $\{\mathcal{Q}_\sigma \colon \sigma \in T\}$ are easily checked to satisfy (A0)-(A7). ■

Proof of Theorem 2 Let $T \subseteq \omega^{<\omega}$, $\{\mathcal{V}_\sigma \colon \sigma \in T\}$, $\{J_n \colon n \in \omega\}$, and $\{\mathcal{Q}_\sigma \colon \sigma \in T\}$ be as in Lemma 23.

Claim 1 For every component $\mathcal{Q}$ of $\mathcal{S}$ there is a $g \in T^\dagger$ such that $\mathcal{Q} = \bigcap_{n\in\omega} \mathcal{V}_{g|n}$ and $\lim_{n\in\omega} \mathcal{V}_{g|n} = \mathcal{Q}$.

Proof. To define g we let $g = \bigcup\{\sigma \in T\colon \mathcal{Q} \subseteq \mathcal{V}_\sigma\}$. It follows from (A3) and (A4) that $g \in T^\dagger$.

Clearly, $\mathcal{Q} \subseteq \bigcap_{n\in\omega} \mathcal{V}_{g|n}$. Consider the sequence $\{\mathcal{Q}_{g|n}\}_{n\in\omega}$.

Suppose $\{\mathcal{Q}_{g|n}\}_{n\in\omega}$ is eventually constant. Let $S \in \bigcap_{n\in\omega} \mathcal{V}_{g|n}$. By (A5), $\mathrm{H}(\mathcal{V}_{g|n}, \mathcal{Q}_{g|n}) < 1/2^n$. It follows that there is a sequence $\{S_n\}_{n\in\omega}$ such that $S_n \in \mathcal{Q}_{g|n}$ and $\lim S_n = S$. There is an $N \in \omega$ such that $\mathcal{Q}_{g|k} = \mathcal{Q}_{g|N}$ for all $k \geq N$. By (A5) and (A7), $\mathcal{Q}_N = \lim \mathcal{Q}_{g|k} = \mathcal{Q}$. Since $\mathcal{Q}$ is closed and $S_k \in \mathcal{Q}$ for almost all k, we have $S \in \mathcal{Q}$. So, $\mathcal{Q} = \bigcap_{n\in\omega} \mathcal{V}_{g|n}$. By (A3), for all $k \geq N$ we have $\mathrm{H}(\mathcal{V}_{g|k}, \mathcal{Q}) = \mathrm{H}(\mathcal{V}_{g|k}, \mathcal{Q}_{g|k}) < 1/2^k$. So, $\lim_{n\in\omega} \mathcal{V}_{g|n} = \mathcal{Q}$.

If $\{\mathcal{Q}_{g|n}\}_{n\in\omega}$ is not eventually constant, then conditions (A6) and (A7) together with Lemma 22 imply that that $\lim \mathrm{diam}(\mathcal{V}_{g|n}) = 0$. So, $\mathcal{Q}$ is a singleton and $\bigcap_{n\in\omega} \mathcal{V}_{g|n} = \mathcal{Q}$ and $\lim_{n\in\omega} \mathcal{V}_{g|n} = \mathcal{Q}$. ■

Claim 2 There are functions $\{h_\sigma\colon \sigma \in T\}$ and points $\{S_\sigma\colon \sigma \in T\}$ so that for every $\sigma, \tau \in T$ we have:

(B1) $h_\sigma\colon \mathcal{Q}_\sigma \to X$ is a continuous selector,

(B2) $S_\sigma \in \mathcal{Q}_\sigma$,

(B3) $S_{\sigma*0} = S_\sigma$ and $h_{\sigma*0} = h_\sigma$, and

(B4) if $|\sigma| > 0$ and τ is the predecessor of σ, then there exists an $S \in \mathcal{Q}_\sigma$ such that $\mathrm{H}(S, S_\tau) < 1/2^{|\tau|}$ and $\mathrm{d}(h_\tau(S_\tau), h_\sigma(S)) < 1/2^{|\tau|}$.

Moreover, we may asssume that $p \in \bigcup_{\sigma\in T} h_\sigma[\mathcal{Q}_\sigma]$.

Proof. By Lemma 20 and (A0) there is a continuous selector $h_\emptyset\colon \mathcal{Q}_\emptyset \to X$ such that $p \in h_\emptyset[\mathcal{Q}_\emptyset]$. Let $S_\emptyset \in Q_\emptyset$ be arbitrary. Clearly these choices satisfy (B1)-(B4).

Assume we have defined h_σ for all $\sigma \in \bigcup_{k=0}^n T_k$ so that (B1)-(B4) are satisfied. Let $\rho \in T_{n+1}$ and $\sigma \in T_n$ be such that $\sigma \subseteq \rho$. If $\rho = \sigma*0$, then, by (A7), we may let $h_\rho = h_\sigma$ and $S_\rho = S_\sigma$. Suppose now that $\rho \neq \sigma*0$. Since $\mathcal{Q}_\rho \subseteq \mathcal{V}_\rho \subseteq \mathcal{V}_\sigma$ and $\mathrm{H}(\mathcal{V}_\sigma, Q_\sigma) < 1/2^{|\sigma|}$, there is a $S_\rho \in \mathcal{Q}_\rho$ such that $\underline{\mathrm{H}}(\mathcal{Q}_\sigma, S_\rho) < 1/2^{|\sigma|}$. Let $S \in \mathcal{Q}_\sigma$ be such that $\mathrm{H}(S, S_\rho) < 1/2^{|\sigma|}$. By the definition of the Hausdorff metric, there is an $x \in S_\rho$ such that $\mathrm{d}(x, h_\sigma(S)) < 1/2^{|\sigma|}$. By Lemma 20, there is a continuous selector $h_\rho\colon \mathcal{Q}_\rho \to X$ such that $h_\rho(S_\rho) = x$. Thus, $S \in \mathcal{Q}_\sigma$ and $\mathrm{d}(h_\rho(S_\rho), h_\sigma(S)) < 1/2^{|\sigma|}$ and $\mathrm{H}(S, S_\rho) < 1/2^{|\sigma|}$. So, we have (B3) and (B4). Clearly, (B1) and (B2) are satisfied by h_ρ and $\mathcal{Q}_\rho$.

By induction, we have that $\{h_\sigma\colon \sigma \in T\}$ and $\{S_\sigma\colon \sigma \in T\}$ satisfy (B1)-(B4). Obviously, $p \in \bigcup_{\sigma\in T} h_\sigma[\mathcal{Q}_\sigma]$. ■

Let h^* be the partial function defined by $h^* = \bigcup_{\sigma\in T} h_\sigma$. Notice that h^* is well defined by (B3), (A2), and (A3).

Claim 3 Let $\sigma \in T_n$. If $R \in \mathcal{V}_\sigma$ and R is in the domain of h^*, then there is a $P \in \mathcal{Q}_\sigma$ such that $\mathrm{d}(h^*(P), h^*(R)) < \sum_{l=n}^\infty 1/2^{l-2}$ and $\mathrm{H}(P, R) < \sum_{l=n}^\infty 1/2^{l-2}$.

Proof. Let $\tau \in T_n$ be such that $R \in \mathcal{Q}_\tau$. By (A7) and (A2), we may assume that $\sigma \subseteq \tau$.

Suppose $0 = |\tau| - |\sigma|$. Let $P = R$ and observe that

$$\max\{\mathrm{d}(h^*(P), h^*(R)), \mathrm{H}(P, R)\} = 0 < \sum_{l=n}^{n+0} 1/2^{l-2}.$$

Assume now that $m \geq 0$ and that we have shown that for every R, if $R \in \mathcal{Q}_\tau$ and $m = |\tau| - |\sigma|$, then there is a $P \in \mathcal{Q}_\sigma$ such that

$$\max\{\mathrm{d}(h^*(P), h^*(R)), \mathrm{H}(P, R)\} < \sum_{l=n}^{n+m} 1/2^{l-2}.$$

We now extend the statement to $m+1$. Let $m+1 = |\tau| - |\sigma|$ and $\sigma \subseteq \tau$. Let ρ be the predecessor of τ. We consider two exhaustive cases.

Suppose $\tau = \rho * 0$. By (A7), $R \in \mathcal{Q}_\rho$. So, by inductive hypothesis there is a $P \in \mathcal{Q}_\sigma$ such that

$$\max\{\mathrm{d}(h^*(P), h^*(R)), \mathrm{H}(P, R)\} < \sum_{l=n}^{n+m} 1/2^{l-2} < \sum_{l=n}^{n+m+1} 1/2^{l-2}.$$

Suppose $\tau \neq \rho * 0$. By (A7) and (A6), $\mathcal{Q}_\tau \notin J_{n+m}$. By (B4), there is an $S \in \mathcal{Q}_\rho$ such that $\mathrm{d}(h^*(S_\tau), h^*(S)) < 1/2^{n+m}$ and $\mathrm{H}(S_\tau, S) < 1/2^{n+m}$. Since $\mathcal{Q}_\tau \notin J_{n+m}$, Lemma 22 implies that $\mathrm{diam}(\mathcal{Q}_\tau) \leq 1/2^{n+m}$. So,

$$\mathrm{H}(R, S) < 1/2^{n+m} + 1/2^{n+m} = 1/2^{n+m-1}.$$

Since $\mathcal{Q}_\tau \notin J_{n+m}$, $\mathrm{diam}(h^*[\mathcal{Q}_\tau]) < 1/2^{m+n}$. So,

$$\mathrm{d}(h^*(R), h^*(S)) < 1/2^{n+m} + 1/2^{n+m} = 1/2^{n+m-1}.$$

By inductive hypothesis, there is a $P \in \mathcal{Q}_\sigma$ such that

$$\max\{\mathrm{H}(P, S), \mathrm{d}(h^*(P), h^*(S))\} < \sum_{l=n}^{n+m} 1/2^{l-2}.$$

Thus,

$$\max\{\mathrm{H}(R, P), \mathrm{d}(h^*(R), h^*(P))\} < \sum_{l=n}^{n+m} 1/2^{l-2} + 1/2^{n+m-1} = \sum_{l=n}^{n+m+1} 1/2^{l-2}.$$

By induction, for every $R \in \mathcal{V}_\sigma$ such that R is in the domain of h^* there is a $P \in \mathcal{Q}_\sigma$ such that $\max\{\mathrm{H}(R, P), \mathrm{d}(h^*(R), h^*(P))\} < \sum_{l=n}^{\infty} 1/2^{l-2}$. ∎

Claim 4 For every $S \in \mathcal{S}$ and $\epsilon > 0$ there is an open neighborhood $\mathcal{U}$ of S such that $\mathrm{diam}(h^*[\mathcal{U}]) < \epsilon$.

Proof. We consider two exhaustive cases.

Case 1 S is in the domain of h^*.
Let $\mathcal{Q}$ be the component of S in $\mathcal{S}$. Notice that $\mathcal{Q}$ is contained in the domain of h^* and $h^*|\mathcal{Q}$ is continuous. Let $\delta > 0$ be such that $\mathrm{d}(h^*(S), h^*(P)) < \epsilon/2$ for all P such that $\mathrm{H}(P, S) < \delta$ and $P \in \mathcal{Q}$. There is a $\sigma \in T$ such that $\mathcal{Q} = \mathcal{Q}_\sigma$ and $|\sigma| = n$ where n is large enough that $\sum_{l=n}^{\infty} 1/2^{l-2} < \min\{\delta/2, \epsilon/2\}$. Let $\mathcal{U}$ be the intersection of the an open neighborhood of S with diameter $\delta/2$ and $\mathcal{V}_\sigma$. Let $R \in \mathcal{U}$ be in the domain of h^*. By Claim 3, there is a $P \in \mathcal{Q}$ such that

$$\max\{\mathrm{H}(R, P), \mathrm{d}(h^*(R), h^*(P))\} < \sum_{l=n}^{\infty} 1/2^{l-2} < \min\{\delta/2, \epsilon/2\}.$$

Since $\mathrm{H}(R, P) < \delta/2$ and $\mathrm{diam}(\mathcal{U}) < \delta/2$, $\mathrm{H}(P, S) < \delta$. By our choice of δ, $\mathrm{d}(h^*(P), h^*(S)) < \epsilon/2$. Since $\mathrm{d}(h^*(R), h^*(P)) < \epsilon/2$, we have $\mathrm{d}(h^*(R), h^*(S)) < \epsilon$.

Case 2 S is not in the domain of h^*.
Let $\mathcal{Q}$ be the component of S in $\mathcal{S}$. Since $\mathcal{Q} \notin \{\mathcal{Q}_\tau : \tau \in T\}$, it follows from (A6) that $\mathcal{Q} \notin \bigcup_{k=1}^{\infty} J_k$. In particular, $\mathcal{Q} = \{S\}$ and S is totally disconnected. By (A4) and Claim 1, there is a $\sigma \in T$ such that $\mathcal{Q} \subseteq \mathcal{V}_\sigma$, no component of $\bigcup \mathcal{Q}_\sigma$ has diameter greater than $\epsilon/3$, and $|\sigma| = n$ where n is large enough that $\sum_{l=n}^{\infty} 1/2^{l-2} < \epsilon/3$. Let $R, P \in \mathcal{V}_\sigma$ be in the domain of h^*. By Claim 3, there exist $G_P, G_R \in \mathcal{Q}_\sigma$ such that

$$\max\{\mathrm{H}(h^*(P), h^*(G_P)), \mathrm{H}(h^*(R), h^*(G_R))\} < \epsilon/3.$$

Since no component of $\mathcal{Q}_\sigma$ has diameter larger than $\epsilon/3$, $\mathrm{d}(h^*(G_P), h^*(G_R)) < \epsilon/3$. Thus, $\mathrm{H}(h^*(P), h^*(R)) < \epsilon$. ■

Since the domain of h^* is dense in $\mathcal{S}$, Claim 3 implies that h^* may be extended to a continuous function h defined on all of $\mathcal{S}$. The continuity of h and the fact that h^* is a selector on its domain implies that h is a selector. By Claim 3, $p \in h[\mathcal{S}]$. ■

References

[1] F. Jordan, S_4 continua in the sense of Michael are dendrites, *Topology Appl.*, **129** (2003), 19–27.

[2] F. Jordan, S. B. Nadler, Jr., A result about a selection problem of Michael, *Proc. Amer. Math. Soc.*, **129**(4) (2001), 1219–1228.

[3] A. Lelek, On the topology of curves, *Fund. Math,.*, **70** (1971), 131–138.

[4] E. Michael, Topologies on spaces of subsets, *Trans. Amer. Math. Soc.*, **71**(1951), (152–182)

[5] S. B. Nadler, Jr., *Continuum Theory*, Marcel Dekker inc. New York and Basel and Hong Kong 1992.

[6] T. Nishiura E.D. and Tymchatyn, Hereditarily Locally connected spaces, *Houston J. Math.*, **2**(4)(1976), (581–599)

[7] Gail S. Young, Lecture Notes in Mathematics, SUNY Buffalo, vol. 171, Springer-Verlag New York, New York 1969.

[8] Gordon Thomas Whyburn, Jr., *Analytic Topology*, Amer. Math .Soc. Collo. Publ., vol.28, Amer. Math. Soc., Providence, RI, 1942.