MEANS ON CHAINABLE CONTINUA

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Abstract. By a mean on a space X we understand a mapping $\mu : X \times X \to X$ such that $\mu(x, y) = \mu(y, x)$ and $\mu(x, x) = x$ for $x, y \in X$. A chainable continuum is a metric compact connected space which admits an ε- mapping onto the interval $[0, 1]$ for every number $\varepsilon > 0$. We show that every chainable continuum that admits a mean is homeomorphic to the interval. In this way we answer a question by P. Bacon. We answer some other question concerning means as well.

A continuum is a metric compact connected space. A continuum X is chainable if for every number $\varepsilon > 0$ there exists an ε-mapping of X onto an interval. This is equivalent to the existence of a representation of X as an inverse sequence of arcs (the joining mappings may be supposed to be surjective). A mapping f of a continuum X onto a continuum Y is said to be weakly confluent if for every continuum $Z \subset Y$ there exists continuum $W \subset X$ such that $f(W) = Z$. A mapping $\mu : X \times X \to X$, where X is a space, is called a mean on X if for every $x, y \in X$ we have $\mu(x, y) = \mu(y, x)$, and $\mu(x, x) = x$. Cohomology means here the Alexander-Čech cohomology. P. Bacon in [B] showed that the $\sin \frac{1}{x}$-curve, one of the standard examples of chainable continua, does not admit a mean and asked whether the only chainable continuum with a mean is the arc. Many works give a partial (positive) answer to the problem. A survey of these results can be found in [Ch]. In this note we give a complete answer to this problem. Our argument uses the idea of K. Sigmon, who in [Sig] applied Alexander-Čech cohomology for investigations of compact spaces admitting a mean. On the other hand our proof is similar to the proof that the only psudocontractible (in the sense of W. Kuperberg) chainable continuum is an arc in [So].

These results were presented at the Henryk Toruńczyk's Seminar at Warsaw University in October 2005. Recently I obtained a preprint by Alejandro Illanes and Hugo Villanueva who solved independently the problem of P. Bacon by different methods.

Let us remind the following standard fact (see [Ch]).

Theorem 1. *Let X and Y be spaces with means μ_X and μ_Y respectively. Then the formula $\mu((x, y), (x', y')) = (\mu_X(x, x'), \mu_Y(y, y'))$ for $(x, y), (x', y') \in X \times Y$ defines a mean on $X \times Y$.*

Proof. We have $\mu((x, y)(x', y')) = (\mu_X(x, x'), \mu_Y(y, y')) = (\mu_X(x', x), \mu_Y(y', y)) = \mu((x', y'), (x, y))$ and $\mu((x, y)(x, y)) = (\mu_X(x, x), \mu_Y(y, y)) = (x, y)$. □

Definition 1. *The mean on $X \times Y$ we described in Theorem 1 will be called* a product mean *and denoted by $\mu_{X\times Y}$.*

Proposition 1. *If $\mu_{X\times X}$ is the product mean on the cartesian square $X \times X$ then $\mu_{X\times X}(\Delta \times \Delta) = \Delta$, where $\Delta = \{(x,x) \in X \times X : \text{ for } x \in X\}$.*

Proof. Let $(x,x),(y,y) \in \Delta$. Then $\mu_{X\times X}((x,x),(y,y)) = (\mu_X(x,y), \mu_X(x,y)) \in \Delta$. □

Definition 2. *Let ε be a positive number. The set $\{(x,y) \in X \times X : \rho(x,y) \leq \varepsilon\}$ we will denote Δ_ε.*

The compactness of a continuum X and Proposition 1 imply the following

Proposition 2. *If X is a continuum with a mean μ_X, then for every number $\varepsilon > 0$ we can choose a number $\delta > 0$ such that $\mu_{X\times X}(\Delta_\delta) \subseteq \Delta_\varepsilon$.*

We have the following

Lemma 1 (Long fold lemma)**.** *Let X be a chainable continuum. If X is not locally connected then there exists a number $\delta > 0$ such that for every number $\varepsilon > 0$ there exist mappings $p : X \to T$, $r : T \to W$, where T, W are arcs, and a subcontinuum $Y \subset X$ satisfying the following properties:*

(i) $\operatorname{diam} Y > \delta$
(ii) *$rp : X \to W$ is an ε-mapping*
(iii) *the sets $L = p(Y)$ and $J = rp(Y)$ are arcs*
(iv) *the mapping $r \mid L : L \to J$ is open and there exist three different arcs L_1, L_2, L_3 each of which is mapped homeomorphically onto J (Fig.1 is a graph of such a map).*

Proof. The non-locally connected continuum X contains a sequence of pairwise disjoint continua K, K_1, K_2, $K_3, \ldots$, such that $\lim K_i = K$ ([Ku], 6.§49.VI. Th.1). Put $\delta = \frac{\operatorname{diam} K}{10}$. Now let us consider a representation of X as an inverse sequence of copies of the unit interval $X = \varprojlim\{I_n,\ f_n^m\}$, where f_n^m are surjections. For a given number $\varepsilon > 0$ let j be such that $f_j : X \twoheadrightarrow I_j$, the projection of the inverse limit is a $\frac{\min(\varepsilon,\delta)}{10}$-mapping. Let $N > 0$ be an integer such that $\operatorname{diam} {f_j}^{-1}([\frac{i}{N}, \frac{i+1}{N}]) \leq \frac{\min(\varepsilon,\delta)}{5}$. Let M be an integer such that for $m \geq M$ the Hausdorff distance between $f_j(K)$ and $f_j(K_m)$ is less than $\frac{1}{4N}$. Now, let $f_k : X \twoheadrightarrow I_k$, $k > j$ be a projection such that the images $f_k(K_M)$, $f_k(K_{M+1}),\ldots$, $f_k(K_{M+4N+4})$ are mutually disjoint and let $i_0 < i_1 \leq N$ be indices such that $\inf(\rho(x,y) : x \in {f_j}^{-1}(\frac{i_0}{N}), y \in {f_j}^{-1}(\frac{i_1}{N})) > \delta$ and $\{\frac{i_0}{N}, \frac{i_1}{N}\} \subseteq f_j(K_m)$, for every $m > M$. Now we define a sequence $a_1 \leq b_1 < c_1 \leq d_1 < a_2 \cdots < a_{2N} \leq b_{2N} < c_{2N} \leq d_{2N} < a_{2N+1}$ of points of the segment I_k. We assume that the continua $K_M, K_{M+1}, \ldots, K_{M+4N+4}$ are indexed in accordance with the order in I_k of their images under the mapping f_k. The point a_1 is the first point I_k belonging to $f_k(K_M)$ for which $f_j^k(a_1) = \frac{i_1}{N}$, the point c_1 is the first point following a_1 such

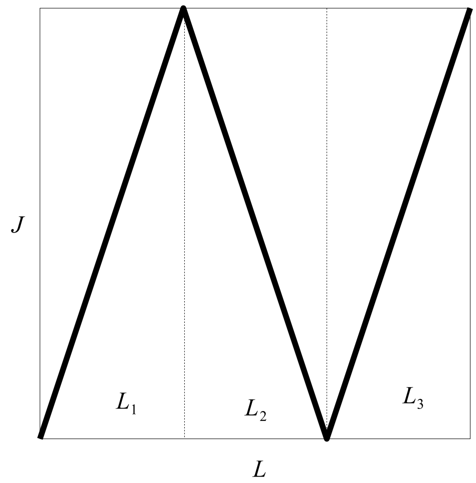

FIGURE 1

that $f_j^k(c_1) = \frac{i_0}{N}$, b_1 is a point of absolute maximum of f_j^k over $[a_1, c_1]$, a_2 is the first point following c_1 such that $f_j^k(a_2) = \frac{i_1}{N}$, d_1 is the point of absolute minimum of f_j^k over $[c_1, a_2]$, etc. After the n-th step of the construction the arc $f_k(K_{M+2n+1})$ lies behind $[a_1, a_n]$ so we can continue the construction. Let $n(x)$ for $x \in I_k$ be the integer fulfilling $f_j^k(x) \in (\frac{n(x)}{N}, \frac{n(x)+1}{N}]$. Each strictly monotone sequence of $n(b_i)$ is shorter than N, hence there is an index l fulfilling $n(b_l) \geq n(b_{l+1}) \leq n(b_{l+2})$. We can assume that $f_j^k(d_l) \geq f_j^k(d_{l+1})$ (otherwise we can invert the order of I_k). Let $q_0 : I_j \to I_j$ be a nondecreasing surjection, which maps the interval $[\frac{n(b_{l+1})}{N}, \frac{n(b_{l+1})+1}{N}]$, to a point, and this interval is the only fiber of this mapping different from a one-point set. Let $q = q_0 f_j^k$. Let us remark that qf_j is an ε-mapping and that $q(b_l) \geq q(b_{l+1}) \leq q(b_{l+2})$ and $q(d_l) \geq q(d_{l+1})$, and $q(b_n) > q(d_m)$ for all indices n, m. Now d_{l+1} is a point of absolute minimum and b_l is a point of absolute maximum of q over $[a_l, a_{l+2}]$. Let us define a nondecreasing sequence of five real numbers: $z_1 = b_l, z_2 = b_l + q(b_l) - q(d_l), z_3 = z_2 + q(b_{l+1}) - q(d_l), z_4 = z_3 + q(b_{l+1}) - q(d_{l+1}), z_5 = z_4 + 1 - d_{l+1}$. We define mappings $p_0 : I \to [0, z_5]$ and $r : [0, z_5] \to I$ by the following formulae:

$$p_0(t) = \begin{cases} t, & \text{for } t \in [0, b_l); \\ z_1 + q(b_l) - q(t), & \text{for } t \in [b_l, d_l); \\ z_2 + q(t) - q(d_l), & \text{for } t \in [d_l, b_{l+1}); \\ z_3 + q(b_{l+1}) - q(t), & \text{for } t \in [b_{l+1}, d_{l+1}); \\ z_4 + t - d_{l+1}, & \text{for } t \in [d_{l+1}, 1]. \end{cases}$$

and

$$r(z) = \begin{cases} q(z), & \text{for } z \in [0, z_1); \\ z_1 + q(b_l) - z, & \text{for } z \in [z_1, z_2); \\ z - z_2 + q(d_l), & \text{for } z \in [z_2, z_3); \\ z_3 + q(b_{l+1}) - z, & \text{for } z \in [z_3, z_4); \\ q(z - z_4 + d_{l+1}), & \text{for } z \in [z_4, z_5]. \end{cases}$$

One can easily check that $q = rp_0$. Let $p = p_0 f_k$. We can put $L_1 = [z_1 + q(b_l) - q(b_{l+1}), z_2]$, $L_2 = [z_2, z_3]$, $L_3 = [z_3, z_3 + q(b_{l+1}) - q(d_l)]$, and $L = L_1 \cup L_2 \cup L_3$, $T = [0, z_5]$, $W = I_j$. Each mapping of a continuum onto an arc is weakly confluent [Bi], hence there exists a continuum $Y \subset X$, such that $p(Y) = L$. □

Lemma 2. *Let $f : X \to S$ be a mapping between compacta . Assume that there exists a mapping $\nu : X \times X \to S$ fulfilling the following conditions: $\nu(x, y) = \nu(y, x)$ and $\nu(x, x) = f(x)$ for $x, y \in X$. Then the induced homomorphism between cohomology groups $f^* : H^1(Y, \mathbb{Z}_2) \to H^1(X, \mathbb{Z}_2)$ must be zero.*

Proof. We have the following commuting diagram

$$\begin{tikzcd}
X \times X \arrow[dd, "\sigma"'] \arrow[drr, "\tilde f"] & & \\
& X \arrow[ul, hook', "\varDelta"'] \arrow[dl, hook, "\varDelta"] \arrow[r, "f"] & S \\
X \times X, \arrow[urr, "\tilde f"'] & &
\end{tikzcd}$$

in which $\sigma : X \times X \to X \times X$ denotes the permutation homeomorphism defined by $\sigma((x, y)) = (y, x)$ and $\varDelta : X \to X \times X$ is given by $\varDelta(x) = (x, x)$. It induces a commuting diagram of cohomology modules of the form

$$\begin{tikzcd}
H^1(X \times X, \mathbb{Z}_2) \arrow[dr, "\varDelta^*"'] & & \\
& H^1(X, \mathbb{Z}_2) & H^1(S, \mathbb{Z}_2) \arrow[l, "f^*"'] \arrow[ull, "\tilde f^*"'] \arrow[dll, "\tilde f^*"] \\
H^1(X \times X, \mathbb{Z}_2), \arrow[uu, "\sigma^*"] \arrow[ur, "\varDelta^*"] & &
\end{tikzcd}$$

.

From the Künneth formula, taking into acount that $\mathbb{Z}_2$ is a field, and hence any module over it is torsion free (cf.[Sig]), we infer that $H^1(X \times X, \mathbb{Z}_2)$ is isomorphic to $H^0(X, \mathbb{Z}_2) \bigotimes H^1(X, \mathbb{Z}_2) \bigoplus H^1(X, \mathbb{Z}_2) \bigotimes H^0(X, \mathbb{Z}_2)$. For convenience we will identify this two objects. For an element $g \in H^1(S^1)$ we have a unique decomposition $\tilde{f}^*(g) = v + w$, where $v \in H^0(X, \mathbb{Z}_2) \bigotimes H^1(X, \mathbb{Z}_2)$ and $w \in H^1(X, \mathbb{Z}_2) \bigotimes H^0(X, \mathbb{Z}_2)$. The components v and w are sums of elements of the form $e_0 \otimes e_1$ and $e_1' \otimes e_0'$ respectively, where $e_0, e_0' \in H^0(X, \mathbb{Z}_2)$ and $e_1, e_1' \in H^1(X, \mathbb{Z}_2)$. We have $\sigma^*(e_0 \otimes e_1) = e_1 \otimes e_0$ and $\sigma^*(e_1' \otimes e_0') = e_0' \otimes e_1'$([Sp]). This means that $\tilde{f}^*(g) = \sigma^* \tilde{f}^*(g)$ is a sum of elements of the form $e_0 \otimes e_1 + e_1 \otimes e_0$. From the diagram $f^*(g) = \Delta^*(\tilde{f}^*(g))$ and $\Delta^*(e_0 \otimes e_1) + \Delta^*(e_1 \otimes e_0) = \Delta^*(e_0 \otimes e_1) + \Delta^*(\sigma^*(e_1 \otimes e_0)) = \Delta^*(e_0 \otimes e_1) + \Delta^*(e_0 \otimes e_1) = 0$, hence f^* is 0. □

Proposition 3. *If X is an acyclic continuum, and $a, b \in X$ and $f : X \to [0,1]$ is a mapping such that $f(a) = 0, f(b) = 1$ then the homomorphism $(f \times f)^* : H^*(([0,1], \{0,1\}) \times ([0,1], \{0,1\}, \mathbb{Z}_2)) \to H^*((X, \{a,b\}) \times (X, \{a,b\}), \mathbb{Z}_2)$ induced by the mapping $f \times f : (X, \{a,b\}) \times (X, \{a,b\}) \to ([0,1], \{0,1\}) \times ([0,1], \{0,1\})$ is an isomorphism.*

Proof. First from the functoriality of the exact sequence for a pair and the Five Isomorphisms Lemma we infer that $f^* : H^*(([0,1], \{0,1\}), \mathbb{Z}_2) \to H^*((X, \{a,b\}), \mathbb{Z}_2)$ is an isomorphism. Then we apply the same reasoning to the Künneth formula for $(f \times f)^*$. □

Theorem 2. *If a chainable continuum X admits a mean then X is an arc.*

Proof. Suppose X is a chainable continuum which is not an arc and $\mu : X \times X \to X$ is a mean. Let $\delta > 0$ be such as in the Long Fold Lemma and let $\eta = 0.1\delta$. From the Proposition 2. we have a number $\varepsilon > 0$ and smaller then η such that $\mu_{X \times X}(\Delta_\varepsilon \times \Delta_\varepsilon) \subset \Delta_\eta$. Now let mappings $p : X \to T$, $r : T \to W$, where T, W are arcs, and a subcontinuum $Y \subset X$ satisfy conditions (i), (ii), (iii) and (iv) of the thesis of the Long Fold Lemma. Because $\operatorname{diam} Y > \delta$ and rp is an ε-mapping there exist points $s, t \in Y$ such that $p(s), p(t) \in L_2$, where L_1, L_2, L_3 are such as in the condition (iv) of the Lemma, and $\rho(s,t) > 0.5\delta$, thus $(p(s), p(t)) \notin p \times p(G_\eta)$. Let us consider the set $G = \{(x,y) \in L \times L : r(x) = r(y)\}$,(see fig.2). It contains a simple closed curve S surrounding $(p(s), p(t))$ (in the figure it is drawn with thick line). Let $d : T \times T \setminus \{(p(s), p(t))\} \to S$ be a retraction. Denote $S' = (p \times p)^{-1}(S)$. Of course $S' \subset G_\eta$. Define $\nu : S' \times S' \to S$ by the formula $\nu(x,y) = d(p \times p(\mu_{X \times X}(x,y)))$, for $x, y \in S'$. We have $\nu(x,y) = \nu(y,x)$, and $\nu(x,x) = p \times p(x)$, hence by lemma 2 the induced homomorphism $(p \times p|S')^* : H^1(S, \mathbb{Z}_2) \to H^1(S', \mathbb{Z}_2)$ must be zero. But on the other hand consider diagram of homomorphisms induced by appropriate restrictions of the mapping $p \times p$ between two cohomology Mayer-Vietoris sequences. One of them is the sequence for pairs $(D, \emptyset), (E, K), (Q, K) = (D, \emptyset) \cup (E, K)$, where D is the "rectangle" bounded by S, $E = \operatorname{cl}(T \times T \setminus D)$, a, b are the ends of T, $K = \{a,b\} \times T \cup T \times \{a,b\}$, and $(Q, K) = (T, \{a,b\}) \times (T, \{a,b\})$. The second is the sequence for pairs

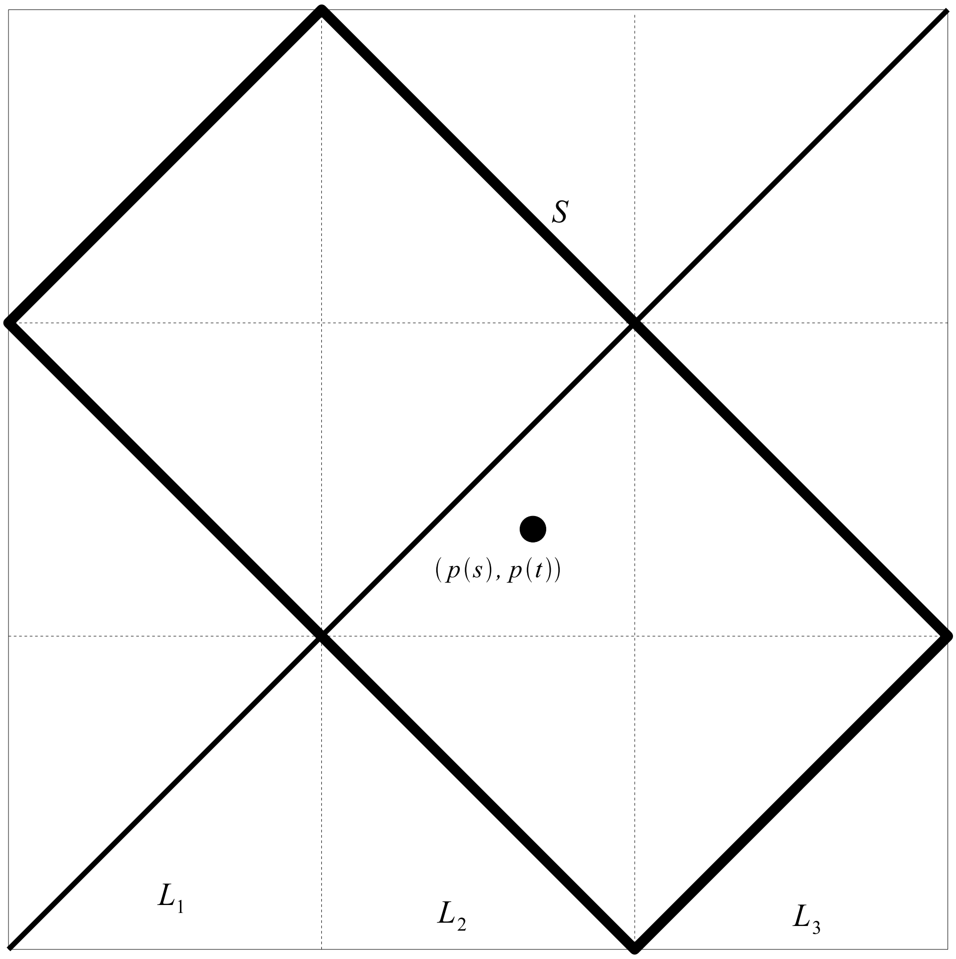

FIGURE 2. The set G.

$(D', \emptyset), (E', K'), (Q', K') = (D', \emptyset) \cup (E', K')$, where D', E' are inverse images under $p \times p$ of D, E respectively, $a', b' \in X$, are such that $p(a') = a, p(b') = b$ and $(Q', K') = (X, \{a', b'\}) \times (X, \{a', b'\})$. Remark that $S = D \cap E$ and $S' = D' \cap E'$. Let us write down the following fragment of this diagram.

$$\begin{array}{ccccccc}
H^1(Q', K') & \longrightarrow & H^1(D', \emptyset) \bigoplus H^1(E', K') & \longrightarrow & H^1(S', \emptyset) & \longrightarrow & H^2(Q', K') \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
H^1(Q, K) & \longrightarrow & H^1(D, \emptyset) \bigoplus H^1(E, K) & \longrightarrow & H^1(S, \emptyset) & \longrightarrow & H^2(Q, K)
\end{array}$$

(for brevity we omitted the ring of coefficients, which is $\mathbb{Z}_2$). From the Proposition 3 the last vertical arrow represents an isomorphism, and the direct sum in the lower row is zero (D is contractible and the pair (E, K) is homotopically equivalent to the pair (K, K)). From exactness of the lower row and commutativity of the diagram the homomorphism from $H^1(S, \mathbb{Z}_2)$ to $H^1(S', \mathbb{Z}_2)$ must be nonzero. A contradiction.

$\square$

Example. Another question Bacon asked in [B] is the following one. Is the arc the only continuum containing an open dense ray (i.e, a subspace homeomorphic to the half-line $[0, \infty)$) that admits a mean? As a counterexample to this question may serve the space Σ' described below. The diadic solenoid Σ as a topological group admits a $1-1$ homomorphism $\varphi : \mathbb{R} \to \Sigma$ of the group of real numbers onto the composant of the neutral element in Σ. As it was remarked by Sigmon in [Sig] solenoid Σ has the unique division by 2. Let us define $\Sigma' = \Sigma \times \{0\} \cup \{(\varphi(t), \frac{1}{t+1}) : t \in [0, \infty)\} \subset \Sigma \times \mathbb{R}$. Now we can define a mean on Σ' by the following fomulae.

i) $\mu((x, z_1), (y, z_2)) = (\frac{x+y}{2}, 0)$ if $z_1 = 0$ or $z_2 = 0$ and $x, y \in \Sigma$;

ii) $\mu((\varphi(t_1), \frac{1}{t_1+1}), (\varphi(t_2), \frac{1}{t_2+1})) = (\varphi(\frac{t_1+t_2}{2}), \frac{1}{\frac{t_1+t_2}{2}+1})$ for $t_1, t_2 \in [0, \infty)$.

Remark. A generalization of means are n-means.

Definition 3. *Let X be a space and n an integer, $n \geq 2$. A mapping $\mu : X^n \to X$ is called an n-mean if $\mu(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = \mu(x_1, \ldots, x_n)$ for every $x_1, \ldots, x_n \in X$ and each permutation σ of indices $1, \ldots, n$, and $\mu(x, \ldots, x) = x$ for every $x \in X$.*

A slight modification of presented argument of Theorem 2 (as a ring of coefficients of cohomology we should use $\mathbb{Z}_p$ instead of $\mathbb{Z}_2$) allows us to show that for any prime integer p the only chainable continuum admitting a p-mean is the arc. But, as was observed by Sigmon [Sig] if a space admits an n-mean then it admits a p-mean for each prime divisor p of the integer n. Hence we have the following general

Theorem 3. *Let $n \geq 2$ be an integer. A chainable continuum X admits an n-mean if and only if X is an arc.*

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