Introduction and Basic Implementation for Finite Element Methods

Chapter 5: Finite elements for 2D steady linear elasticity equation

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Outline

Outline

Target problem

- Consider the 2D linear elasticity equation:

$$\begin{cases} -\nabla \cdot \sigma(\mathbf{u}) = \mathbf{f} \qquad in \quad \Omega, \\ \mathbf{u} = \mathbf{g} \qquad on \quad \partial\Omega. \end{cases}$$

where

$$\mathbf{u}(x_1, x_2) = (u_1,\ u_2)^t,\ \mathbf{g}(x_1, x_2) = (g_1,\ g_2)^t,\ \mathbf{f}(x_1, x_2) = (f_1,\ f_2)^t.$$

- The stress tensor $\sigma(\mathbf{u})$ is defined as

$$\sigma(\mathbf{u}) = \begin{pmatrix} \sigma_{11}(\mathbf{u}) & \sigma_{12}(\mathbf{u}) \\ \sigma_{21}(\mathbf{u}) & \sigma_{22}(\mathbf{u}) \end{pmatrix},\ \sigma_{ij}(\mathbf{u}) = \lambda\,(\nabla\cdot\mathbf{u})\delta_{ij} + 2\mu\epsilon_{ij}(\mathbf{u}),$$

where λ and μ are Lam$\acute{e}$ parameters.

Target problem

- The strain tensor is defined as

$$\epsilon = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{pmatrix}, \qquad \epsilon_{ij}(\mathbf{u}) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big),$$

where

$$\delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

- Hence the stress tensor can be written as

$$\sigma(\mathbf{u}) = \begin{pmatrix} \lambda\frac{\partial u_1}{\partial x_1} + 2\mu\frac{\partial u_1}{\partial x_1} + \lambda\frac{\partial u_2}{\partial x_2} & \mu\frac{\partial u_1}{\partial x_2} + \mu\frac{\partial u_2}{\partial x_1} \\ \mu\frac{\partial u_1}{\partial x_2} + \mu\frac{\partial u_2}{\partial x_1} & \lambda\frac{\partial u_1}{\partial x_1} + \lambda\frac{\partial u_2}{\partial x_2} + 2\mu\frac{\partial u_2}{\partial x_2} \end{pmatrix}$$

Weak formulation

- First, take the inner product with a vector function $\mathbf{v}(x_1, x_2) = (v_1,\ v_2)^t$ on both sides of the original equation:

$$
\begin{aligned}
& -\nabla \cdot \sigma(\mathbf{u}) = \mathbf{f} \quad \text{in } \Omega \\
\Rightarrow \quad & -(\nabla \cdot \sigma(\mathbf{u})) \cdot \mathbf{v} = \mathbf{f} \cdot \mathbf{v} \quad \text{in } \Omega \\
\Rightarrow \quad & -\int_\Omega (\nabla \cdot \sigma(\mathbf{u})) \cdot \mathbf{v}\ dx_1 dx_2 = \int_\Omega \mathbf{f} \cdot \mathbf{v}\ dx_1 dx_2.
\end{aligned}
$$

- $\mathbf{u}(x_1, x_2)$ is called a trail function and $\mathbf{v}(x_1, x_2)$ is called a test function.

Weak formulation

- Second, using integration by parts in multi-dimension:

$$\int_{\Omega} (\nabla \cdot \sigma(\mathbf{u})) \cdot \mathbf{v} \; dx_1 dx_2 = \int_{\partial\Omega} (\sigma(\mathbf{u})\mathbf{n}) \cdot \mathbf{v} \; ds - \int_{\Omega} \sigma(\mathbf{u}) : \nabla \mathbf{v} \; dx_1 dx_2,$$

where $\mathbf{n} = (n_1, n_2)^t$ is the unit outer normal vector of $\partial\Omega$, we obtain

$$\int_{\Omega} \sigma(\mathbf{u}) : \nabla \mathbf{v} \; dx_1 dx_2 - \int_{\partial\Omega} (\sigma(\mathbf{u})\mathbf{n}) \cdot \mathbf{v} \; ds = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \; dx_1 dx_2.$$

Here,

$$\begin{aligned} A : B &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ &= a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}, \end{aligned}$$

Weak formulation

- and

$$\nabla \mathbf{v} = \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} \end{pmatrix}.$$

- Since the solution on the domain boundary $\partial\Omega$ are given by $\mathbf{u} = \mathbf{g}$, then we can choose the test function $\mathbf{v}(x_1, x_2)$ such that $\mathbf{v} = 0$ on $\partial\Omega$.

- Hence

$$\int_\Omega \sigma(\mathbf{u}) : \nabla \mathbf{v} \; dx_1 dx_2 = \int_\Omega \mathbf{f} \cdot \mathbf{v} \; dx_1 dx_2.$$

Weak formulation

- Weak formulation in the vector format: find $\mathbf{u} \in H^1(\Omega) \times H^1(\Omega)$ such that

$$\int_\Omega \sigma(\mathbf{u}) : \nabla \mathbf{v} \; dx_1 dx_2 = \int_\Omega \mathbf{f} \cdot \mathbf{v} \; dx_1 dx_2$$

for any $\mathbf{v} \in H_0^1(\Omega) \times H_0^1(\Omega)$.

- Let $a(\mathbf{u}, \mathbf{v}) = \int_\Omega \sigma(\mathbf{u}) : \nabla \mathbf{v} \; dx_1 dx_2$ and $(\mathbf{f}, \mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} \; dx_1 dx_2$.

- Weak formulation: find $\mathbf{u} \in H^1(\Omega) \times H^1(\Omega)$ such that

$$a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v})$$

for any $\mathbf{v} \in H_0^1(\Omega) \times H_0^1(\Omega)$.

Weak formulation

- In details,

$$
\begin{aligned}
& \sigma(\mathbf{u}) : \nabla \mathbf{v} \\
= & \begin{pmatrix} \sigma_{11}(\mathbf{u}) & \sigma_{12}(\mathbf{u}) \\ \sigma_{21}(\mathbf{u}) & \sigma_{22}(\mathbf{u}) \end{pmatrix} : \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} \end{pmatrix} \\
= & \sigma_{11}(\mathbf{u}) \frac{\partial v_1}{\partial x_1} + \sigma_{12}(\mathbf{u}) \frac{\partial v_1}{\partial x_2} + \sigma_{21}(\mathbf{u}) \frac{\partial v_2}{\partial x_1} + \sigma_{22}(\mathbf{u}) \frac{\partial v_2}{\partial x_2} \\
= & \left(\lambda \frac{\partial u_1}{\partial x_1} + 2\mu \frac{\partial u_1}{\partial x_1} + \lambda \frac{\partial u_2}{\partial x_2} \right) \frac{\partial v_1}{\partial x_1} \\
& + \left(\mu \frac{\partial u_1}{\partial x_2} + \mu \frac{\partial u_2}{\partial x_1} \right) \frac{\partial v_1}{\partial x_2} + \left(\mu \frac{\partial u_1}{\partial x_2} + \mu \frac{\partial u_2}{\partial x_1} \right) \frac{\partial v_2}{\partial x_1} \\
& + \left(\lambda \frac{\partial u_1}{\partial x_1} + \lambda \frac{\partial u_2}{\partial x_2} + 2\mu \frac{\partial u_2}{\partial x_2} \right) \frac{\partial v_2}{\partial x_2}
\end{aligned}
$$

Weak formulation

- Then

$$
\begin{aligned}
&\int_\Omega \sigma(\mathbf{u}) : \nabla \mathbf{v} \; dx_1 dx_2 \\
= \quad &\int_\Omega \Big(\lambda \frac{\partial u_1}{\partial x_1}\frac{\partial v_1}{\partial x_1} + 2\mu \frac{\partial u_1}{\partial x_1}\frac{\partial v_1}{\partial x_1} + \lambda \frac{\partial u_2}{\partial x_2}\frac{\partial v_1}{\partial x_1} \\
&+\mu \frac{\partial u_1}{\partial x_2}\frac{\partial v_1}{\partial x_2} + \mu \frac{\partial u_2}{\partial x_1}\frac{\partial v_1}{\partial x_2} + \mu \frac{\partial u_1}{\partial x_2}\frac{\partial v_2}{\partial x_1} + \mu \frac{\partial u_2}{\partial x_1}\frac{\partial v_2}{\partial x_1} \\
&+\lambda \frac{\partial u_1}{\partial x_1}\frac{\partial v_2}{\partial x_2} + \lambda \frac{\partial u_2}{\partial x_2}\frac{\partial v_2}{\partial x_2} + 2\mu \frac{\partial u_2}{\partial x_2}\frac{\partial v_2}{\partial x_2} \Big) \; dx_1 dx_2.
\end{aligned}
$$

- Also, we have

$$
\int_\Omega \mathbf{f} \cdot \mathbf{v} \; dx_1 dx_2 = \int_\Omega (f_1 v_1 + f_2 v_2) \; dx_1 dx_2.
$$

Weak formulation

- Weak formulation in the scalar format: find $u_1 \in H^1(\Omega)$ and $u_2 \in H^1(\Omega)$ such that

$$
\begin{aligned}
& \int_\Omega \Big(\lambda \frac{\partial u_1}{\partial x_1}\frac{\partial v_1}{\partial x_1} + 2\mu \frac{\partial u_1}{\partial x_1}\frac{\partial v_1}{\partial x_1} + \lambda \frac{\partial u_2}{\partial x_2}\frac{\partial v_1}{\partial x_1} \\
& +\mu \frac{\partial u_1}{\partial x_2}\frac{\partial v_1}{\partial x_2} + \mu \frac{\partial u_2}{\partial x_1}\frac{\partial v_1}{\partial x_2} + \mu \frac{\partial u_1}{\partial x_2}\frac{\partial v_2}{\partial x_1} + \mu \frac{\partial u_2}{\partial x_1}\frac{\partial v_2}{\partial x_1} \\
& +\lambda \frac{\partial u_1}{\partial x_1}\frac{\partial v_2}{\partial x_2} + \lambda \frac{\partial u_2}{\partial x_2}\frac{\partial v_2}{\partial x_2} + 2\mu \frac{\partial u_2}{\partial x_2}\frac{\partial v_2}{\partial x_2}\Big)\, dx_1 dx_2 \\
= \quad & \int_\Omega (f_1 v_1 + f_2 v_2)\, dx_1 dx_2.
\end{aligned}
$$

for any $v_1 \in H_0^1(\Omega)$ and $v_2 \in H_0^1(\Omega)$.

Galerkin formulation

- Assume there is a finite dimensional subspace $U_h \subset H^1(\Omega)$. Define U_{h0} to be the space which consists of the functions of U_h with value 0 on the Dirichlet boundary.
- Then the Galerkin formulation is to find $\mathbf{u}_h \in U_h \times U_h$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h)$$
$$\Leftrightarrow \quad \int_\Omega \sigma(\mathbf{u}_h) : \nabla \mathbf{v}_h \; dx_1 dx_2 = \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \; dx_1 dx_2$$

for any $\mathbf{v}_h \in U_{h0} \times U_{h0}$.
- Basic idea of Galerkin formulation: use finite dimensional space to approximate infinite dimensional space.
- Here $U_h = span\{\phi_j\}_{j=1}^{N_b}$ is chosen to be a finite element space where $\{\phi_j\}_{j=1}^{N_b}$ are the global finite element basis functions, such as those defined in Chapter 2.

Galerkin formulation

- For an easier implementation, we use the following Galerkin formulation (without considering the Dirichlet boundary condition, which will be handled later): find $\mathbf{u}_h \in U_h \times U_h$ such that

$$
\begin{aligned}
& a(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) \\
\Leftrightarrow \quad & \int_\Omega \sigma(\mathbf{u}_h) : \nabla \mathbf{v}_h \; dx_1 dx_2 = \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \; dx_1 dx_2
\end{aligned}
$$

for any $\mathbf{v}_h \in U_h \times U_h$.

Galerkin formulation

- In details, the Galerkin formulation (without considering the Dirichlet boundary condition, which will be handled later) is to find $u_{1h} \in U_h$ and $u_{2h} \in U_h$ such that

$$\begin{aligned}
&\int_\Omega \Big(\lambda \frac{\partial u_{1h}}{\partial x_1}\frac{\partial v_{1h}}{\partial x_1} + 2\mu \frac{\partial u_{1h}}{\partial x_1}\frac{\partial v_{1h}}{\partial x_1} + \lambda \frac{\partial u_{2h}}{\partial x_2}\frac{\partial v_{1h}}{\partial x_1} \\
&+\mu \frac{\partial u_{1h}}{\partial x_2}\frac{\partial v_{1h}}{\partial x_2} + \mu \frac{\partial u_{2h}}{\partial x_1}\frac{\partial v_{1h}}{\partial x_2} + \mu \frac{\partial u_{1h}}{\partial x_2}\frac{\partial v_{2h}}{\partial x_1} + \mu \frac{\partial u_{2h}}{\partial x_1}\frac{\partial v_{2h}}{\partial x_1} \\
&+\lambda \frac{\partial u_{1h}}{\partial x_1}\frac{\partial v_{2h}}{\partial x_2} + \lambda \frac{\partial u_{2h}}{\partial x_2}\frac{\partial v_{2h}}{\partial x_2} + 2\mu \frac{\partial u_{2h}}{\partial x_2}\frac{\partial v_{2h}}{\partial x_2}\Big)\, dx_1 dx_2 \\
= &\int_\Omega (f_1 v_{1h} + f_2 v_{2h})\, dx_1 dx_2.
\end{aligned}$$

for any $v_{1h} \in U_h$ and $v_{2h} \in U_h$.

Outline

Discretization formulation

Recall the following definitions from Chapter 2:

- N: number of mesh elements.
- N_m: number of mesh nodes.
- E_n $(n = 1, \cdots, N)$: mesh elements.
- Z_k $(k = 1, \cdots, N_m)$: mesh nodes.
- N_l: number of local mesh nodes in a mesh element.
- P:information matrix consisting of the coordinates of all mesh nodes.
- T: information matrix consisting of the global node indices of the mesh nodes of all the mesh elements.

Discretization formulation

- We only consider the nodal basis functions (Lagrange type) in this course.
- N_{lb}: number of local finite element nodes (=number of local finite element basis functions) in a mesh element.
- N_b: number of the finite element nodes ($=$ the number of unknowns $=$ the total number of the finite element basis functions).
- X_j $(j = 1, \cdots, N_b)$: finite element nodes.
- P_b: information matrix consisting of the coordinates of all finite element nodes.
- T_b: information matrix consisting of the global node indices of the finite element nodes of all the mesh elements.

Discretization formulation

- Since $u_{1h},\ u_{2h} \in U_h = span\{\phi_j\}_{j=1}^{N_b}$, then
$$u_{1h} = \sum_{j=1}^{N_b} u_{1j}\phi_j, \quad u_{2h} = \sum_{j=1}^{N_b} u_{2j}\phi_j$$
for some coefficients u_{1j} and u_{2j} $(j = 1, \cdots, N_b)$.
- If we can set up a linear algebraic system for u_{1j} and u_{2j} $(j = 1, \cdots, N_b)$, then we can solve it to obtain the finite element solution $\mathbf{u}_h = (u_{1h}, u_{2h})^t$.
- We choose $\mathbf{v}_h = (\phi_i, 0)^t$ $(i = 1, \cdots, N_b)$ and $\mathbf{v}_h = (0, \phi_i)^t$ $(i = 1, \cdots, N_b)$ in the Galerkin formulation. That is, in the first set of test functions, we choose $v_{1h} = \phi_i$ $(i = 1, \cdots, N_b)$ and $v_{2h} = 0$; in the second set of test functions, we choose $v_{1h} = 0$ and $v_{2h} = \phi_i$ $(i = 1, \cdots, N_b)$.

Discretization formulation

- Set $\mathbf{v}_h = (\phi_i, 0)^t$, i.e., $v_{1h} = \phi_i$ and $v_{2h} = 0$ $(i = 1, \cdots, N_b)$. Then

$$\int_\Omega \lambda \left(\sum_{j=1}^{N_b} u_{1j} \frac{\partial \phi_j}{\partial x_1} \right) \frac{\partial \phi_i}{\partial x_1} dx_1 dx_2 + 2 \int_\Omega \mu \left(\sum_{j=1}^{N_b} u_{1j} \frac{\partial \phi_j}{\partial x_1} \right) \frac{\partial \phi_i}{\partial x_1} dx_1 dx_2$$

$$+ \int_\Omega \lambda \left(\sum_{j=1}^{N_b} u_{2j} \frac{\partial \phi_j}{\partial x_2} \right) \frac{\partial \phi_i}{\partial x_1} dx_1 dx_2 +$$

$$\int_\Omega \mu \left(\sum_{j=1}^{N_b} u_{1j} \frac{\partial \phi_j}{\partial x_2} \right) \frac{\partial \phi_i}{\partial x_2} dx_1 dx_2 + \int_\Omega \mu \left(\sum_{j=1}^{N_b} u_{2j} \frac{\partial \phi_j}{\partial x_1} \right) \frac{\partial \phi_i}{\partial x_2} dx_1 dx_2$$

$$= \int_\Omega f_1 \phi_i dx_1 dx_2.$$

Discretization formulation

- Set $\mathbf{v}_h = (0, \phi_i)^t$, i.e., $v_{1h} = 0$ and $v_{2h} = \phi_i$ $(i = 1, \cdots, N_b)$. Then

$$\int_\Omega \mu \left(\sum_{j=1}^{N_b} u_{1j} \frac{\partial \phi_j}{\partial x_2} \right) \frac{\partial \phi_i}{\partial x_1} dx_1 dx_2 + \int_\Omega \mu \left(\sum_{j=1}^{N_b} u_{2j} \frac{\partial \phi_j}{\partial x_1} \right) \frac{\partial \phi_i}{\partial x_1} dx_1 dx_2$$

$$+ \int_\Omega \lambda \left(\sum_{j=1}^{N_b} u_{1j} \frac{\partial \phi_j}{\partial x_1} \right) \frac{\partial \phi_i}{\partial x_2} dx_1 dx_2$$

$$+ \int_\Omega \lambda \left(\sum_{j=1}^{N_b} u_{2j} \frac{\partial \phi_j}{\partial x_2} \right) \frac{\partial \phi_i}{\partial x_2} dx_1 dx_2$$

$$+2 \int_\Omega \mu \left(\sum_{j=1}^{N_b} u_{2j} \frac{\partial \phi_j}{\partial x_2} \right) \frac{\partial \phi_i}{\partial x_2} dx_1 dx_2$$

$$= \int_\Omega f_2 \phi_i dx_1 dx_2.$$

Discretization formulation

- Simplify the above two sets of equations, we obtain

$$
\begin{aligned}
&\sum_{j=1}^{N_b} u_{1j} \left(\int_\Omega \lambda \frac{\partial \phi_j}{\partial x_1} \frac{\partial \phi_i}{\partial x_1} dx_1 dx_2 + 2 \int_\Omega \mu \frac{\partial \phi_j}{\partial x_1} \frac{\partial \phi_i}{\partial x_1} dx_1 dx_2 + \int_\Omega \mu \frac{\partial \phi_j}{\partial x_2} \frac{\partial \phi_i}{\partial x_2} dx_1 dx_2 \right) \\
&+ \sum_{j=1}^{N_b} u_{2j} \left(\int_\Omega \lambda \frac{\partial \phi_j}{\partial x_2} \frac{\partial \phi_i}{\partial x_1} dx_1 dx_2 + \int_\Omega \mu \frac{\partial \phi_j}{\partial x_1} \frac{\partial \phi_i}{\partial x_2} dx_1 dx_2 \right) \\
&= \int_\Omega f_1 \phi_i dx_1 dx_2 \\
&\sum_{j=1}^{N_b} u_{1j} \left(\int_\Omega \lambda \frac{\partial \phi_j}{\partial x_1} \frac{\partial \phi_i}{\partial x_2} dx_1 dx_2 + \int_\Omega \mu \frac{\partial \phi_j}{\partial x_2} \frac{\partial \phi_i}{\partial x_1} dx_1 dx_2 \right) \\
&+ \sum_{j=1}^{N_b} u_{2j} (\int_\Omega \lambda \frac{\partial \phi_j}{\partial x_2} \frac{\partial \phi_i}{\partial x_2} dx_1 dx_2 + 2 \int_\Omega \mu \frac{\partial \phi_j}{\partial x_2} \frac{\partial \phi_i}{\partial x_2} dx_1 dx_2 \\
&+ \int_\Omega \mu \frac{\partial \phi_j}{\partial x_1} \frac{\partial \phi_i}{\partial x_1} dx_1 dx_2) \\
&= \int_\Omega f_2 \phi_i dx_1 dx_2.
\end{aligned}
$$

Matrix formulation

- Define

$$A_1 = \left[\int_\Omega \lambda \frac{\partial \phi_j}{\partial x_1}\frac{\partial \phi_i}{\partial x_1} dx_1 dx_2\right]_{i,j=1}^{N_b}, A_2 = \left[\int_\Omega \mu \frac{\partial \phi_j}{\partial x_1}\frac{\partial \phi_i}{\partial x_1} dx_1 dx_2\right]_{i,j=1}^{N_b},$$

$$A_3 = \left[\int_\Omega \mu \frac{\partial \phi_j}{\partial x_2}\frac{\partial \phi_i}{\partial x_2} dx_1 dx_2\right]_{i,j=1}^{N_b}, A_4 = \left[\int_\Omega \lambda \frac{\partial \phi_j}{\partial x_2}\frac{\partial \phi_i}{\partial x_1} dx_1 dx_2\right]_{i,j=1}^{N_b},$$

$$A_5 = \left[\int_\Omega \mu \frac{\partial \phi_j}{\partial x_1}\frac{\partial \phi_i}{\partial x_2} dx_1 dx_2\right]_{i,j=1}^{N_b}, A_6 = \left[\int_\Omega \lambda \frac{\partial \phi_j}{\partial x_1}\frac{\partial \phi_i}{\partial x_2} dx_1 dx_2\right]_{i,j=1}^{N_b},$$

$$A_7 = \left[\int_\Omega \mu \frac{\partial \phi_j}{\partial x_2}\frac{\partial \phi_i}{\partial x_1} dx_1 dx_2\right]_{i,j=1}^{N_b}, A_8 = \left[\int_\Omega \lambda \frac{\partial \phi_j}{\partial x_2}\frac{\partial \phi_i}{\partial x_2} dx_1 dx_2\right]_{i,j=1}^{N_b}.$$

- Each matrix above can be obtained by Algorithm I-3 in Chapter 3.
- Then

$$A = \begin{pmatrix} A_1 + 2A_2 + A_3 & A_4 + A_5 \\ A_6 + A_7 & A_8 + 2A_3 + A_2 \end{pmatrix}$$

Matrix formulation

- Define the load vector

$$\vec{b} = \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \end{pmatrix}$$

where

$$\vec{b}_1 = \left[\int_\Omega f_1 \phi_i dx_1 dx_2 \right]_{i=1}^{N_b}, \quad \vec{b}_2 = \left[\int_\Omega f_2 \phi_i dx_1 dx_2 \right]_{i=1}^{N_b}.$$

- Each of $\vec{b}_1$ and $\vec{b}_2$ can be obtained by Algorithm II-3 in Chapter 3.

Matrix formulation

- Define the unknown vector

$$\vec{X} = \begin{pmatrix} \vec{X}_1 \\ \vec{X}_2 \end{pmatrix}$$

where

$$\vec{X}_1 = [u_{1j}]_{j=1}^{N_b}, \quad \vec{X}_2 = [u_{2j}]_{j=1}^{N_b}.$$

- Then we obtain the linear algebraic system

$$A\vec{X} = \vec{b}.$$

Outline

Dirichlet boundary condition

- Basically, the Dirichlet boundary condition $\mathbf{u} = \mathbf{g}$ (i.e., $u_1 = g_1$ and $u_2 = g_2$) provides the solutions at all boundary finite element nodes.
- Since the coefficient u_{1j} and u_{2j} in the finite element solutions $u_{1h} = \sum_{j=1}^{N_b} u_{1j}\phi_j$ and $u_{2h} = \sum_{j=1}^{N_b} u_{2j}\phi_j$ are actually the numerical solutions at the finite element node X_j $(j = 1, \cdots, N_b)$ when nodal basis functions are used, we actually know those u_{1j} and u_{2j} which are corresponding to the boundary finite element nodes.
- Recall that boundarynodes(2,:) store the global node indices of all boundary finite element nodes.
- If $m \in \ boundarynodes(2,:)$, then the m^{th} equation is called a boundary node equation for u_1 and the $(N_b + m)^{th}$ equation is called a boundary node equation for u_2.
- Set nbn to be the number of boundary nodes;

Dirichlet boundary condition

- One way to impose the Dirichlet boundary condition is to replace the boundary node equations in the linear system by the following equations

$$
\begin{aligned}
u_{1m} &= g_1(X_m) \\
u_{2m} &= g_2(X_m).
\end{aligned}
$$

for all $m \in \ boundarynodes(2,:)$.

This is similar to $u_m = g(X_m)$ in Chapter 3.

Dirichlet boundary condition

Based on Algorithm III in Chapter 3, we obtain Algorithm III-3:

- Deal with the Dirichlet boundary conditions:

  $FOR\ k = 1, \cdots, nbn$:

  If $boundarynodes(1,k)$ shows Dirichlet condition, then

  $i = boundarynodes(2,k)$;

  $A(i,:) = 0$;

  $A(i,i) = 1$;

  $b(i) = g_1(P_b(:,i))$;

  $A(N_b + i,:) = 0$;

  $A(N_b + i, N_b + i) = 1$;

  $b(N_b + i) = g_2(P_b(:,i))$;

  $ENDIF$

  END

Outline

Universal framework of the finite element method

Recall from Chapter 3:

- Generate the mesh information: matrices P and T;
- Assemble the matrices and vectors: local assembly based on P and T only;
- Deal with the boundary conditions: boundary information matrix and local assembly;
- Solve linear systems: numerical linear algebra.

Algorithm

Recall from Chapter 3:

- Generate the mesh information matrices P and T.
- Assemble the stiffness matrix A by using Algorithm I. (We will choose Algorithm I-3 in class)
- Assemble the load vector $\vec{b}$ by using Algorithm II. (We will choose Algorithm II-3 in class)
- Deal with the Dirichlet boundary condition by using Algorithm III-3.
- Solve $A\vec{X} = \vec{b}$ for $\vec{X}$ by using a direct or iterative method.

Algorithm

Recall Algorithm I-3 from Chapter 3:

- Initialize the matrix: $A = sparse(N_b^{test}, N_b^{trial})$;
- Compute the integrals and assemble them into A:

$FOR\ n = 1, \cdots, N$

$\quad FOR\ \alpha = 1, \cdots, N_{lb}^{trial}$

$\quad\quad FOR\ \beta = 1, \cdots, N_{lb}^{test}$

$\quad\quad\quad$ Compute $r = \int_{E_n} c \frac{\partial^{r+s}\varphi_{n\alpha}}{\partial x^r \partial y^s} \frac{\partial^{p+q}\psi_{n\beta}}{\partial x^p \partial y^q}\, dxdy$;

$\quad\quad\quad$ Add r to $A(T_b^{test}(\beta, n), T_b^{trial}(\alpha, n))$.

$\quad\quad END$

$\quad END$

END

Algorithm

- Call Algorithm I-3 with $r = 1,\ s = 0,\ p = 1,$ and $q = 0$ and $c = \lambda$ to obtain A_1.
- Call Algorithm I-3 with $r = 1,\ s = 0,\ p = 1,$ and $q = 0$ and $c = \mu$ to obtain A_2.
- Call Algorithm I-3 with $r = 0,\ s = 1,\ p = 0,$ and $q = 1$ and $c = \mu$ to obtain A_3.
- Call Algorithm I-3 with $r = 0,\ s = 1,\ p = 1,$ and $q = 0$ and $c = \lambda$ to obtain A_4.
- Call Algorithm I-3 with $r = 1,\ s = 0,\ p = 0,$ and $q = 1$ and $c = \mu$ to obtain A_5.
- Call Algorithm I-3 with $r = 1,\ s = 0,\ p = 0,$ and $q = 1$ and $c = \lambda$ to obtain A_6.
- Call Algorithm I-3 with $r = 0,\ s = 1,\ p = 1,$ and $q = 0$ and $c = \mu$ to obtain A_7.
- Call Algorithm I-3 with $r = 0,\ s = 1,\ p = 0,$ and $q = 1$ and $c = \lambda$ to obtain A_8.
- Then the stiffness matrix
  $A = [A_1 + 2A_2 + A_3 \quad A_4 + A_5; A_6 + A_7 \quad A_8 + 2A_3 + A_2]$.

Algorithm

Recall Algorithm II-3 from Chapter 3:

- Initialize the vector: $b = sparse(N_b, 1)$;
- Compute the integrals and assemble them into b:

$FOR\ n = 1, \cdots, N$:

$\quad FOR\ \beta = 1, \cdots, N_{lb}$:

$\quad\quad$ Compute $r = \int_{E_n} f \frac{\partial^{p+q} \psi_{n\beta}}{\partial x^p \partial y^q}\, dx_1 dx_2$;

$\quad\quad b(T_b(\beta, n), 1) = b(T_b(\beta, n), 1) + r$;

$\quad END$

END

Algorithm

- Call Algorithm II-3 with $p = q = 0$ and $f = f_1$ to obtain b_1.
- Call Algorithm II-3 with $p = q = 0$ and $f = f_2$ to obtain b_2.
- Then the load vector $\vec{b} = [b_1; b_2]$.

Algorithm

Recall Algorithm III-3 from this Chapter:

- Deal with the Dirichlet boundary conditions:

$FOR\ k = 1, \cdots, nbn$:

If $boundarynodes(1, k)$ shows Dirichlet condition, then

$i = boundarynodes(2, k)$;

$A(i, :) = 0$;

$A(i, i) = 1$;

$b(i) = g_1(P_b(:, i))$;

$A(N_b + i, :) = 0$;

$A(N_b + i, N_b + i) = 1$;

$b(N_b + i) = g_2(P_b(:, i))$;

$ENDIF$

END

Measurements for errors

- L^∞ norm error:

$$\|\mathbf{u}-\mathbf{u}_h\|_\infty = \max\left(\|u_1-u_{1h}\|_\infty, \|u_2-u_{2h}\|_\infty\right),$$
$$\|u_1-u_{1h}\|_\infty = \sup_\Omega |u_1-u_{1h}|,$$
$$\|u_2-u_{2h}\|_\infty = \sup_\Omega |u_2-u_{2h}|.$$

- L^2 norm error:

$$\|\mathbf{u}-\mathbf{u}_h\|_0 = \sqrt{\|u_1-u_{1h}\|_0^2 + \|u_2-u_{2h}\|_0^2},$$
$$\|u_1-u_{1h}\|_0 = \sqrt{\int_\Omega (u_1-u_{1h})^2 dx_1 dx_2},$$
$$\|u_2-u_{2h}\|_0 = \sqrt{\int_\Omega (u_2-u_{2h})^2 dx_1 dx_2}.$$

Measurements for errors

- H^1 semi-norm error:

$$|\mathbf{u}-\mathbf{u}_h|_1 = \sqrt{|u_1-u_{1h}|_1^2 + |u_2-u_{2h}|_1^2},$$

$$|u_1-u_{1h}|_1 = \sqrt{\int_\Omega \left(\frac{\partial(u_1-u_{1h})}{\partial x_1}\right)^2 + \left(\frac{\partial(u_1-u_{1h})}{\partial x_2}\right)^2 dx_1dx_2},$$

$$|u_2-u_{2h}|_1 = \sqrt{\int_\Omega \left(\frac{\partial(u_2-u_{2h})}{\partial x_1}\right)^2 + \left(\frac{\partial(u_2-u_{2h})}{\partial x_2}\right)^2 dx_1dx_2}.$$

- Basic idea: call Algorithm IV and Algorithm V in Chapter 3 for each of u_1 and u_2; then plug the results into the above formulas for the errors of $\mathbf{u}$.

Numerical example

- Example 1: Use the finite element method to solve the following equation on the domain $\Omega = [0,1] \times [0,1]$:

$$\begin{aligned} -\nabla \cdot \sigma(\mathbf{u}) = \mathbf{f} \quad & \text{on } \Omega, \\ u_1 = 0, u_2 = 0 \quad & \text{on } \partial\Omega, \end{aligned}$$

with

$$\begin{aligned} f_1 &= -(\lambda + 3\mu)(-\pi^2 \sin(\pi x)\sin(\pi y)) \\ &\quad -(\lambda+\mu)((2x-1)(2y-1)), \\ f_2 &= -(\lambda+2\mu)(2x(x-1)) - \\ &\quad (\lambda+\mu)(\pi^2\cos(\pi x)\cos(\pi y)) - \mu(2y(y-1)). \end{aligned}$$

Here $\lambda = 1$ and $\mu = 2$.

Numerical example

- The analytic solution of this problem is $u_1 = \sin(\pi x)\sin(\pi y)$ and $u_2 = x(x-1)y(y-1)$, which can be used to compute the errors of the numerical solution. We can also verify f_1 and f_2 above by plugging the analytic solutions into the elasticity equation.

- Let's code for the linear and quadratic finite element method of the 2D linear elasticity equation together!

- Open your Matlab!

Numerical example

h	$\Vert\mathbf{u}-\mathbf{u}_h\Vert_\infty$	$\Vert\mathbf{u}-\mathbf{u}_h\Vert_0$	$\vert\mathbf{u}-\mathbf{u}_h\vert_1$
$1/8$	5.1175×10^{-2}	2.2934×10^{-2}	4.3382×10^{-1}
$1/16$	1.3250×10^{-2}	5.9217×10^{-3}	2.1821×10^{-1}
$1/32$	3.3437×10^{-3}	1.4938×10^{-3}	1.0926×10^{-1}
$1/64$	8.3793×10^{-4}	3.7431×10^{-4}	5.4649×10^{-2}

Table: The numerical errors for linear finite element.

- Any Observation?
- Second order convergence $O(h^2)$ in L^2/L^∞ norm and first order convergence $O(h)$ in H^1 semi-norm, which match the optimal approximation capability expected from piecewise linear functions.

Numerical example

h	$\|\mathbf{u}-\mathbf{u}_h\|_\infty$	$\|\mathbf{u}-\mathbf{u}_h\|_0$	$\lvert\mathbf{u}-\mathbf{u}_h\rvert_1$
$1/8$	1.4862×10^{-3}	5.0157×10^{-4}	3.3555×10^{-2}
$1/16$	1.8944×10^{-4}	6.2157×10^{-5}	8.4431×10^{-3}
$1/32$	2.3799×10^{-5}	7.7475×10^{-6}	2.1142×10^{-3}
$1/64$	2.9797×10^{-6}	9.6770×10^{-7}	5.2876×10^{-4}

Table: The numerical errors for quadratic finite element.

- Any Observation?
- Third order convergence $O(h^3)$ in L^2/L^∞ norm and second order convergence $O(h^2)$ in H^1 semi-norm, which match the optimal approximation capability expected from piecewise quadratic functions.

Outline

Stress boundary condition

- Consider
$$\begin{cases} -\nabla \cdot \sigma(\mathbf{u}) = \mathbf{f} & in \quad \Omega, \\ \sigma(\mathbf{u})\mathbf{n} = \mathbf{p} & on \quad \partial\Omega. \end{cases}$$
where $\mathbf{n} = (n_1, n_2)^t$ is the unit outer normal vector of $\partial\Omega$ and
$$\mathbf{p}(x_1, x_2) = (p_1, p_2)^t, \ \mathbf{f}(x_1, x_2) = (f_1, f_2)^t.$$
- Recall
$$\int_\Omega \sigma(\mathbf{u}) : \nabla \mathbf{v} \ dx_1 dx_2 - \int_{\partial\Omega} (\sigma(\mathbf{u})\mathbf{n}) \cdot \mathbf{v} \ ds = \int_\Omega \mathbf{f} \cdot \mathbf{v} \ dx_1 dx_2.$$
- Hence
$$\int_\Omega \sigma(\mathbf{u}) : \nabla \mathbf{v} \ dx_1 dx_2 - \int_{\partial\Omega} \mathbf{p} \cdot \mathbf{v} \ ds = \int_\Omega \mathbf{f} \cdot \mathbf{v} \ dx_1 dx_2.$$
- Is there anything wrong? The solution is not unique!

Stress boundary condition

- Recall that

$$\sigma(\mathbf{u}) = \begin{pmatrix} \lambda\frac{\partial u_1}{\partial x_1} + 2\mu\frac{\partial u_1}{\partial x_1} + \lambda\frac{\partial u_2}{\partial x_2} & \mu\frac{\partial u_1}{\partial x_2} + \mu\frac{\partial u_2}{\partial x_1} \\ \mu\frac{\partial u_1}{\partial x_2} + \mu\frac{\partial u_2}{\partial x_1} & \lambda\frac{\partial u_1}{\partial x_1} + \lambda\frac{\partial u_2}{\partial x_2} + 2\mu\frac{\partial u_2}{\partial x_2} \end{pmatrix}$$

- Then, if $\mathbf{u} = (u_1, u_2)^t$ is a solution, then $\mathbf{u} + \mathbf{c}$ is also a solution where $\mathbf{c}$ is a constant vector.

Stress boundary condition

- Consider

$$
\begin{aligned}
-\nabla \cdot \sigma(\mathbf{u}) = \mathbf{f} \qquad & in \quad \Omega, \\
\sigma(\mathbf{u})\mathbf{n} = \mathbf{p} \ \ & \text{on } \Gamma_S \subset \partial\Omega, \\
\mathbf{u} = \mathbf{g} \ \ & \text{on } \Gamma_D = \partial\Omega/\Gamma_S.
\end{aligned}
$$

  where $\mathbf{n} = (n_1, \, n_2)^t$ is the unit outer normal vector of Γ_S.

- Recall

$$
\int_\Omega \sigma(\mathbf{u}) : \nabla \mathbf{v} \; dx_1 dx_2 - \int_{\partial\Omega} (\sigma(\mathbf{u})\mathbf{n}) \cdot \mathbf{v} \; ds = \int_\Omega \mathbf{f} \cdot \mathbf{v} \; dx_1 dx_2.
$$

- Since the solution on $\Gamma_D = \partial\Omega/\Gamma_S$ is given by $\mathbf{u} = \mathbf{g}$, then we can choose the test function $\mathbf{v}(x_1, x_2)$ such that $\mathbf{v} = 0$ on $\partial\Omega/\Gamma_S$.

Stress boundary condition

- Hence

$$\begin{aligned}\int_{\partial\Omega} (\sigma(\mathbf{u})\mathbf{n}) \cdot \mathbf{v}\ ds &= \int_{\Gamma_S} (\sigma(\mathbf{u})\mathbf{n}) \cdot \mathbf{v}\ ds + \int_{\partial\Omega/\Gamma_S} (\sigma(\mathbf{u})\mathbf{n}) \cdot \mathbf{v}\ ds \\ &= \int_{\Gamma_S} \mathbf{p} \cdot \mathbf{v}\ ds,\end{aligned}$$

- The weak formulation is to find $\mathbf{u} \in H^1(\Omega) \times H^1(\Omega)$ such that

$$\int_{\Omega} \sigma(\mathbf{u}) : \nabla \mathbf{v}\ dx_1 dx_2 = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}\ dx_1 dx_2 + \int_{\Gamma_S} \mathbf{p} \cdot \mathbf{v}\ ds,$$

for any $\mathbf{v} \in H^1_{0D}(\Omega) \times H^1_{0D}(\Omega)$. Here

$$\int_{\Gamma_S} \mathbf{p} \cdot \mathbf{v}\ ds = \int_{\Gamma_S} p_1 v_1\ ds + \int_{\Gamma_S} p_2 v_2\ ds,$$

$$H^1_{0D}(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}.$$

Stress boundary condition

- Then the Galerkin formulation is to find $\mathbf{u}_h \in U_h \times U_h$ such that

$$\int_\Omega \sigma(\mathbf{u}_h) : \nabla \mathbf{v}_h \, dx_1 dx_2 = \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \, dx_1 dx_2 + \int_{\Gamma_S} \mathbf{p} \cdot \mathbf{v}_h \, ds.$$

  for any $\mathbf{v}_h \in U_{h0} \times U_{h0}$.

- For an easier implementation, we use the following Galerkin formulation (without considering the Dirichlet boundary condition, which will be handled later): find $\mathbf{u}_h \in U_h \times U_h$ such that

$$\int_\Omega \sigma(\mathbf{u}_h) : \nabla \mathbf{v}_h \, dx_1 dx_2 = \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \, dx_1 dx_2 + \int_{\Gamma_S} \mathbf{p} \cdot \mathbf{v}_h \, ds.$$

  for any $\mathbf{v}_h \in U_h \times U_h$.

Stress boundary condition

- Since $u_{1h},\ u_{2h} \in U_h = span\{\phi_j\}_{j=1}^{N_b}$, then

$$u_{1h} = \sum_{j=1}^{N_b} u_{1j}\phi_j, \quad u_{2h} = \sum_{j=1}^{N_b} u_{2j}\phi_j$$

for some coefficients u_{1j} and u_{2j} $\ (j = 1, \cdots, N_b)$.

- For the test function, we choose $\mathbf{v}_h = (\phi_i, 0)^t$ $(i = 1, \cdots, N_b)$ and $\mathbf{v}_h = (0, \phi_i)^t$ $(i = 1, \cdots, N_b)$.

Stress boundary condition

- Then by the same procedure to derive the matrix formulation before, we obtain the following linear system

$$\sum_{j=1}^{N_b} u_{1j}\left(\int_\Omega \lambda \frac{\partial \phi_j}{\partial x_1}\frac{\partial \phi_i}{\partial x_1}dx_1dx_2 + 2\int_\Omega \mu \frac{\partial \phi_j}{\partial x_1}\frac{\partial \phi_i}{\partial x_1}dx_1dx_2 + \int_\Omega \mu \frac{\partial \phi_j}{\partial x_2}\frac{\partial \phi_i}{\partial x_2}dx_1dx_2\right)$$

$$+\sum_{j=1}^{N_b} u_{2j}\left(\int_\Omega \lambda \frac{\partial \phi_j}{\partial x_2}\frac{\partial \phi_i}{\partial x_1}dx_1dx_2 + \int_\Omega \mu \frac{\partial \phi_j}{\partial x_1}\frac{\partial \phi_i}{\partial x_2}dx_1dx_2\right)$$

$$= \int_\Omega f_1\phi_i dx_1dx_2 + \int_{\Gamma_S} p_1\phi_i\ ds$$

$$\sum_{j=1}^{N_b} u_{1j}\left(\int_\Omega \lambda \frac{\partial \phi_j}{\partial x_1}\frac{\partial \phi_i}{\partial x_2}dx_1dx_2 + \int_\Omega \mu \frac{\partial \phi_j}{\partial x_2}\frac{\partial \phi_i}{\partial x_1}dx_1dx_2\right)$$

$$+\sum_{j=1}^{N_b} u_{2j}(\int_\Omega \lambda \frac{\partial \phi_j}{\partial x_2}\frac{\partial \phi_i}{\partial x_2}dx_1dx_2 + 2\int_\Omega \mu \frac{\partial \phi_j}{\partial x_2}\frac{\partial \phi_i}{\partial x_2}dx_1dx_2$$

$$+\int_\Omega \mu \frac{\partial \phi_j}{\partial x_1}\frac{\partial \phi_i}{\partial x_1}dx_1dx_2)$$

$$= \int_\Omega f_2\phi_i dx_1dx_2 + \int_{\Gamma_S} p_2\phi_i\ ds.$$

Stress boundary condition

- Recall

$$A_1 = \left[\int_\Omega \lambda \frac{\partial \phi_j}{\partial x_1} \frac{\partial \phi_i}{\partial x_1} dx_1 dx_2 \right]_{i,j=1}^{N_b}, A_2 = \left[\int_\Omega \mu \frac{\partial \phi_j}{\partial x_1} \frac{\partial \phi_i}{\partial x_1} dx_1 dx_2 \right]_{i,j=1}^{N_b},$$

$$A_3 = \left[\int_\Omega \mu \frac{\partial \phi_j}{\partial x_2} \frac{\partial \phi_i}{\partial x_2} dx_1 dx_2 \right]_{i,j=1}^{N_b}, A_4 = \left[\int_\Omega \lambda \frac{\partial \phi_j}{\partial x_2} \frac{\partial \phi_i}{\partial x_1} dx_1 dx_2 \right]_{i,j=1}^{N_b},$$

$$A_5 = \left[\int_\Omega \mu \frac{\partial \phi_j}{\partial x_1} \frac{\partial \phi_i}{\partial x_2} dx_1 dx_2 \right]_{i,j=1}^{N_b}, A_6 = \left[\int_\Omega \lambda \frac{\partial \phi_j}{\partial x_1} \frac{\partial \phi_i}{\partial x_2} dx_1 dx_2 \right]_{i,j=1}^{N_b},$$

$$A_7 = \left[\int_\Omega \mu \frac{\partial \phi_j}{\partial x_2} \frac{\partial \phi_i}{\partial x_1} dx_1 dx_2 \right]_{i,j=1}^{N_b}, A_8 = \left[\int_\Omega \lambda \frac{\partial \phi_j}{\partial x_2} \frac{\partial \phi_i}{\partial x_2} dx_1 dx_2 \right]_{i,j=1}^{N_b}.$$

and

$$A = \begin{pmatrix} A_1 + 2A_2 + A_3 & A_4 + A_5 \\ A_6 + A_7 & A_8 + 2A_3 + A_2 \end{pmatrix}$$

Stress boundary condition

- Recall

$$\vec{b} = \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \end{pmatrix}$$

where

$$\vec{b}_1 = \left[\int_\Omega f_1 \phi_i dx_1 dx_2 \right]_{i=1}^{N_b}, \quad \vec{b}_2 = \left[\int_\Omega f_2 \phi_i dx_1 dx_2 \right]_{i=1}^{N_b}.$$

- Recall

$$\vec{X} = \begin{pmatrix} \vec{X}_1 \\ \vec{X}_2 \end{pmatrix}$$

where

$$\vec{X}_1 = [u_{1j}]_{j=1}^{N_b}, \quad \vec{X}_2 = [u_{2j}]_{j=1}^{N_b}.$$

Stress boundary condition

- Define the additional vector from the stress boundary condition:

$$\vec{v} = \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \end{pmatrix}$$

where

$$\vec{v}_1 = \left[\int_{\Gamma_S} p_1 \phi_i \, ds \right]_{i=1}^{N_b}, \quad \vec{v}_2 = \left[\int_{\Gamma_S} p_2 \phi_i \, ds \right]_{i=1}^{N_b}.$$

- Define the new vector $\widetilde{\vec{b}} = \vec{b} + \vec{v}$.
- Then we obtain the linear algebraic system

$$A\vec{X} = \widetilde{\vec{b}}.$$

- Since each of $\vec{v}_1$ and $\vec{v}_2$ is similar to the $\vec{v}$ for the Neumann condition in Chapter 3, we essentially only need repeat the code of Neumman condition in Chapter 3 for $\vec{v}_1$ and $\vec{v}_2$.

Stress boundary condition

Based on Algorithm VI in Chapter 3, we obtain Algorithm VI-2:

- Initialize the vector: $v = sparse(2N_b, 1)$;
- Compute the integrals and assemble them into v:

$FOR\ k = 1, \cdots, nbe$:

  $IF\ boundaryedges(1, k)$ shows stress boundary, $THEN$

    $n_k = boundaryedges(2, k)$;

    $FOR\ \beta = 1, \cdots, N_{lb}$:

      Compute $r = \int_{e_k} p_1 \frac{\partial^{a+b} \psi_{n_k \beta}}{\partial x_1^a \partial x_2^b}\ ds$;

      $v(T_b(\beta, n_k), 1) = v(T_b(\beta, n_k), 1) + r$;

      Compute $r = \int_{e_k} p_2 \frac{\partial^{a+b} \psi_{n_k \beta}}{\partial x_1^a \partial x_2^b}\ ds$;

      $v(N_b + T_b(\beta, n_k), 1) = v(N_b + T_b(\beta, n_k), 1) + r$;

    END

  $ENDIF$

END

Robin boundary conditions

- Consider

$$
\begin{aligned}
-\nabla \cdot \sigma(\mathbf{u}) = \mathbf{f} \qquad &in \quad \Omega, \\
\sigma(\mathbf{u})\mathbf{n} + r\mathbf{u} = \mathbf{q} \ \ &\text{on } \Gamma_R \subseteq \partial\Omega, \\
\mathbf{u} = \mathbf{g} \ \ &\text{on } \Gamma_D = \partial\Omega/\Gamma_R.
\end{aligned}
$$

  where $\mathbf{n} = (n_1, \, n_2)^t$ is the unit outer normal vector of Γ_R.
- Recall

$$
\int_\Omega \sigma(\mathbf{u}) : \nabla \mathbf{v} \; dx_1 dx_2 - \int_{\partial\Omega} (\sigma(\mathbf{u})\mathbf{n}) \cdot \mathbf{v} \; ds = \int_\Omega \mathbf{f} \cdot \mathbf{v} \; dx_1 dx_2.
$$

- Since the solution on $\Gamma_D = \partial\Omega/\Gamma_R$ is given by $\mathbf{u} = \mathbf{g}$, then we can choose the test function $\mathbf{v}(x_1, x_2)$ such that $\mathbf{v} = 0$ on $\partial\Omega/\Gamma_R$.

Robin boundary condition

- Hence

$$
\begin{aligned}
\int_{\partial\Omega} (\sigma(\mathbf{u})\mathbf{n}) \cdot \mathbf{v} \; ds &= \int_{\Gamma_R} (\sigma(\mathbf{u})\mathbf{n}) \cdot \mathbf{v} \; ds + \int_{\partial\Omega/\Gamma_R} (\sigma(\mathbf{u})\mathbf{n}) \cdot \mathbf{v} \; ds \\
&= \int_{\Gamma_R} \mathbf{q} \cdot \mathbf{v} \; ds - \int_{\Gamma_R} r\mathbf{u} \cdot \mathbf{v} \; ds,
\end{aligned}
$$

- The weak formulation is to find $\mathbf{u} \in H^1(\Omega) \times H^1(\Omega)$ such that

$$
\int_\Omega \sigma(\mathbf{u}) : \nabla \mathbf{v} dx_1 dx_2 + \int_{\Gamma_R} r\mathbf{u} \cdot \mathbf{v} ds = \int_\Omega \mathbf{f} \cdot \mathbf{v} dx_1 dx_2 + \int_{\Gamma_R} \mathbf{q} \cdot \mathbf{v} ds,
$$

for any $\mathbf{v} \in H^1_{0D}(\Omega) \times H^1_{0D}(\Omega)$. Here

$$
\int_{\Gamma_R} \mathbf{q} \cdot \mathbf{v} \; ds = \int_{\Gamma_R} q_1 v_1 \; ds + \int_{\Gamma_R} q_2 v_2 \; ds,
$$

$$
\int_{\Gamma_R} r\mathbf{u} \cdot \mathbf{v} \; ds = \int_{\Gamma_R} r u_1 v_1 \; ds + \int_{\Gamma_R} r u_2 v_2 \; ds,
$$

$$
H^1_{0D}(\Omega) = \{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D \}.
$$

Robin boundary condition

- Then the Galerkin formulation is to find $\mathbf{u}_h \in U_h \times U_h$ such that

$$\int_\Omega \sigma(\mathbf{u}_h) : \nabla \mathbf{v}_h \, dx_1 dx_2 + \int_{\Gamma_R} r \mathbf{u}_h \cdot \mathbf{v}_h \, ds = \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \, dx_1 dx_2 + \int_{\Gamma_R} \mathbf{q} \cdot \mathbf{v}_h \, ds.$$

for any $\mathbf{v}_h \in U_{h0} \times U_{h0}$.

- For an easier implementation, we use the following Galerkin formulation (without considering the Dirichlet boundary condition, which will be handled later): find $\mathbf{u}_h \in U_h \times U_h$ such that

$$\int_\Omega \sigma(\mathbf{u}_h) : \nabla \mathbf{v}_h \, dx_1 dx_2 + \int_{\Gamma_R} r \mathbf{u}_h \cdot \mathbf{v}_h \, ds = \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \, dx_1 dx_2 + \int_{\Gamma_R} \mathbf{q} \cdot \mathbf{v}_h \, ds.$$

for any $\mathbf{v}_h \in U_h \times U_h$.

Robin boundary condition

- Since $u_{1h},\ u_{2h} \in U_h = span\{\phi_j\}_{j=1}^{N_b}$, then

$$u_{1h} = \sum_{j=1}^{N_b} u_{1j}\phi_j, \quad u_{2h} = \sum_{j=1}^{N_b} u_{2j}\phi_j$$

  for some coefficients u_{1j} and u_{2j} $\ (j = 1, \cdots, N_b)$.

- For the test function, we choose $\mathbf{v}_h = (\phi_i, 0)^t$ $(i = 1, \cdots, N_b)$ and $\mathbf{v}_h = (0, \phi_i)^t$ $(i = 1, \cdots, N_b)$.

Robin boundary condition

- Then by the same procedure to derive the matrix formulation before, we obtain the following linear system

$$
\begin{aligned}
&\sum_{j=1}^{N_b} u_{1j}\left(\int_\Omega (\lambda+2\mu)\frac{\partial\phi_j}{\partial x_1}\frac{\partial\phi_i}{\partial x_1}dx_1dx_2 + \int_\Omega \mu\frac{\partial\phi_j}{\partial x_2}\frac{\partial\phi_i}{\partial x_2}dx_1dx_2 + \int_{\Gamma_R} r\phi_j\phi_i\ ds\right)\\
&+\sum_{j=1}^{N_b} u_{2j}\left(\int_\Omega \lambda\frac{\partial\phi_j}{\partial x_2}\frac{\partial\phi_i}{\partial x_1}dx_1dx_2 + \int_\Omega \mu\frac{\partial\phi_j}{\partial x_1}\frac{\partial\phi_i}{\partial x_2}dx_1dx_2\right)\\
&=\int_\Omega f_1\phi_i dx_1dx_2 + \int_{\Gamma_S} q_1\phi_i\ ds\\
&\sum_{j=1}^{N_b} u_{1j}\left(\int_\Omega \lambda\frac{\partial\phi_j}{\partial x_1}\frac{\partial\phi_i}{\partial x_2}dx_1dx_2 + \int_\Omega \mu\frac{\partial\phi_j}{\partial x_2}\frac{\partial\phi_i}{\partial x_1}dx_1dx_2\right)\\
&+\sum_{j=1}^{N_b} u_{2j}\left(\int_\Omega (\lambda+2\mu)\frac{\partial\phi_j}{\partial x_2}\frac{\partial\phi_i}{\partial x_2}dx_1dx_2 + \int_\Omega \mu\frac{\partial\phi_j}{\partial x_1}\frac{\partial\phi_i}{\partial x_1}dx_1dx_2 + \int_{\Gamma_R} r\phi_j\phi_i\ ds\right)\\
&=\int_\Omega f_2\phi_i dx_1dx_2 + \int_{\Gamma_S} q_2\phi_i\ ds.
\end{aligned}
$$

Robin boundary condition

- Recall

$$A_1 = \left[\int_\Omega \lambda \frac{\partial \phi_j}{\partial x_1}\frac{\partial \phi_i}{\partial x_1} dx_1 dx_2\right]_{i,j=1}^{N_b}, A_2 = \left[\int_\Omega \mu \frac{\partial \phi_j}{\partial x_1}\frac{\partial \phi_i}{\partial x_1} dx_1 dx_2\right]_{i,j=1}^{N_b},$$

$$A_3 = \left[\int_\Omega \mu \frac{\partial \phi_j}{\partial x_2}\frac{\partial \phi_i}{\partial x_2} dx_1 dx_2\right]_{i,j=1}^{N_b}, A_4 = \left[\int_\Omega \lambda \frac{\partial \phi_j}{\partial x_2}\frac{\partial \phi_i}{\partial x_1} dx_1 dx_2\right]_{i,j=1}^{N_b},$$

$$A_5 = \left[\int_\Omega \mu \frac{\partial \phi_j}{\partial x_1}\frac{\partial \phi_i}{\partial x_2} dx_1 dx_2\right]_{i,j=1}^{N_b}, A_6 = \left[\int_\Omega \lambda \frac{\partial \phi_j}{\partial x_1}\frac{\partial \phi_i}{\partial x_2} dx_1 dx_2\right]_{i,j=1}^{N_b},$$

$$A_7 = \left[\int_\Omega \mu \frac{\partial \phi_j}{\partial x_2}\frac{\partial \phi_i}{\partial x_1} dx_1 dx_2\right]_{i,j=1}^{N_b}, A_8 = \left[\int_\Omega \lambda \frac{\partial \phi_j}{\partial x_2}\frac{\partial \phi_i}{\partial x_2} dx_1 dx_2\right]_{i,j=1}^{N_b},$$

and

$$A = \begin{pmatrix} A_1 + 2A_2 + A_3 & A_4 + A_5 \\ A_6 + A_7 & A_8 + 2A_3 + A_2 \end{pmatrix}$$

Robin boundary condition

- Recall

$$\vec{b} = \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \end{pmatrix}$$

where

$$\vec{b}_1 = \left[\int_\Omega f_1 \phi_i dx_1 dx_2 \right]_{i=1}^{N_b}, \quad \vec{b}_2 = \left[\int_\Omega f_2 \phi_i dx_1 dx_2 \right]_{i=1}^{N_b}.$$

- Recall

$$\vec{X} = \begin{pmatrix} \vec{X}_1 \\ \vec{X}_2 \end{pmatrix}$$

where

$$\vec{X}_1 = [u_{1j}]_{j=1}^{N_b}, \quad \vec{X}_2 = [u_{2j}]_{j=1}^{N_b}.$$

Robin boundary condition

- Define the additional vector from the Robin boundary condition:

$$\vec{w} = \begin{pmatrix} \vec{w}_1 \\ \vec{w}_2 \end{pmatrix}$$

where

$$\vec{w}_1 = \left[\int_{\Gamma_S} q_1 \phi_i \ ds\right]_{i=1}^{N_b}, \quad \vec{w}_2 = \left[\int_{\Gamma_S} q_2 \phi_i \ ds\right]_{i=1}^{N_b}.$$

- Define the new vector $\widetilde{\vec{b}} = \vec{b} + \vec{w}$.
- Since each of $\vec{w}_1$ and $\vec{w}_2$ is similar to the $\vec{w}$ for the Robin condition in Chapter 3, we essentially only need repeat the code of $\vec{w}$ in Chapter 3 for $\vec{w}_1$ and $\vec{w}_2$.

Robin boundary condition

- Define the additional matrix from the Robin boundary condition

$$R = [r_{ij}]_{i,j=1}^{N_b} = \left[\int_{\Gamma_R} r \phi_j \phi_i \, ds \right]_{i,j=1}^{N_b}.$$

- Define the new matrix:

$$\widetilde{A} = \begin{pmatrix} A_1 + 2A_2 + A_3 + R & A_4 + A_5 \\ A_6 + A_7 & A_8 + 2A_3 + A_2 + R \end{pmatrix}$$

- Then we obtain the linear algebraic system

$$\widetilde{A}\vec{X} = \widetilde{\vec{b}}.$$

- Since R is the same as the R in Chapter 3, the code for R is the same. But R needs to be added to the matrix A twice as showed above to obtain $\widetilde{A}$.
- Pseudo code? (Part of a project for you)

Dirichlet/stress/Robin mixed boundary condition

- Consider

$$
\begin{aligned}
&-\nabla\cdot\sigma(\mathbf{u})=\mathbf{f} \qquad in \quad \Omega,\\
&\sigma(\mathbf{u})\mathbf{n}=\mathbf{p} \;\; \text{on } \Gamma_S\subset\partial\Omega,\\
&\sigma(\mathbf{u})\mathbf{n}+r\mathbf{u}=\mathbf{q} \;\; \text{on } \Gamma_R\subseteq\partial\Omega,\\
&\mathbf{u}=\mathbf{g} \;\; \text{on } \Gamma_D=\partial\Omega/(\Gamma_S\cup\Gamma_R).
\end{aligned}
$$

where $\mathbf{n}=(n_1,\, n_2)^t$ is the unit outer normal vector of $\Gamma_S\bigcup\Gamma_R$.

- Recall

$$
\int_\Omega \sigma(\mathbf{u}):\nabla\mathbf{v}\;dx_1dx_2-\int_{\partial\Omega}(\sigma(\mathbf{u})\mathbf{n})\cdot\mathbf{v}\;ds=\int_\Omega \mathbf{f}\cdot\mathbf{v}\;dx_1dx_2.
$$

- Since the solution on $\Gamma_D=\partial\Omega/(\Gamma_S\cup\Gamma_R)$ is given by $\mathbf{u}=\mathbf{g}$, then we can choose the test function $\mathbf{v}(x_1,x_2)$ such that $\mathbf{v}=0$ on $\partial\Omega/(\Gamma_S\cup\Gamma_R)$.

Dirichlet/stress/Robin mixed boundary condition

- Combining the above derivation for stress and Robin boundary conditions, we obtain

$$
\begin{aligned}
& \int_{\partial\Omega} (\sigma(\mathbf{u})\mathbf{n}) \cdot \mathbf{v} \ ds \\
= \ & \int_{\Gamma_S} (\sigma(\mathbf{u})\mathbf{n}) \cdot \mathbf{v} \ ds + \int_{\Gamma_R} (\sigma(\mathbf{u})\mathbf{n}) \cdot \mathbf{v} \ ds \\
& + \int_{\partial\Omega/(\Gamma_S \cup \Gamma_R)} (\sigma(\mathbf{u})\mathbf{n}) \cdot \mathbf{v} \ ds \\
= \ & \int_{\Gamma_S} \mathbf{p} \cdot \mathbf{v} \ ds + \int_{\Gamma_R} \mathbf{q} \cdot \mathbf{v} \ ds - \int_{\Gamma_R} r\mathbf{u} \cdot \mathbf{v} \ ds,
\end{aligned}
$$

Dirichlet/stress/Robin mixed boundary condition

- The weak formulation is to find $\mathbf{u} \in H^1(\Omega) \times H^1(\Omega)$ such that

$$\int_\Omega \sigma(\mathbf{u}) : \nabla \mathbf{v} \; dx_1 dx_2 + \int_{\Gamma_R} r\mathbf{u} \cdot \mathbf{v} \; ds$$
$$= \int_\Omega \mathbf{f} \cdot \mathbf{v} \; dx_1 dx_2 + \int_{\Gamma_R} \mathbf{q} \cdot \mathbf{v} \; ds + \int_{\Gamma_S} \mathbf{p} \cdot \mathbf{v} \; ds.$$

  for any $\mathbf{v} \in H^1_{0D}(\Omega) \times H^1_{0D}(\Omega)$. Here $H^1_{0D}(\Omega) = \{w \in H^1(\Omega) : w = 0 \text{ on } \Gamma_D\}$.
- Code?
- Combine all of the subroutines for Dirichlet/stress/Robin boundary conditions.

Stress boundary condition in normal/tangential directions

- Consider

$$
\begin{aligned}
&-\nabla \cdot \sigma(\mathbf{u}) = \mathbf{f} \qquad in \quad \Omega, \\
&\mathbf{n}^t \sigma(\mathbf{u})\mathbf{n} = p_n, \;\; \tau^t \sigma(\mathbf{u})\mathbf{n} = p_\tau \;\; \text{on } \Gamma_S \subset \partial\Omega, \\
&\mathbf{u} = \mathbf{g} \;\; \text{on } \Gamma_D = \partial\Omega/\Gamma_S.
\end{aligned}
$$

  where $\mathbf{n} = (n_1, n_2)^t$ is the unit outer normal vector of Γ_S and $\tau = (\tau_1, \tau_2)^t$ is the corresponding unit tangential vector of Γ_S.
- Recall

$$
\int_\Omega \sigma(\mathbf{u}) : \nabla \mathbf{v} \; dx_1 dx_2 - \int_{\partial\Omega} (\sigma(\mathbf{u})\mathbf{n}) \cdot \mathbf{v} \; ds = \int_\Omega \mathbf{f} \cdot \mathbf{v} \; dx_1 dx_2.
$$

- Since the solution on $\Gamma_D = \partial\Omega/\Gamma_S$ is given by $\mathbf{u} = \mathbf{g}$, then we can choose the test function $\mathbf{v}(x_1, x_2)$ such that $\mathbf{v} = 0$ on $\partial\Omega/\Gamma_S$.

Stress boundary condition in normal/tangential directions

- Using the above conditions, orthogonal decomposition of a vector, and the definition of unit normal/tangential vector, we obtain

$$
\begin{aligned}
& \int_{\partial\Omega} (\sigma(\mathbf{u})\mathbf{n}) \cdot \mathbf{v} \, ds \\
= \; & \int_{\Gamma_S} (\sigma(\mathbf{u})\mathbf{n}) \cdot \mathbf{v} \, ds + \int_{\partial\Omega/\Gamma_S} (\sigma(\mathbf{u})\mathbf{n}) \cdot \mathbf{v} \, ds \\
= \; & \int_{\Gamma_S} (\sigma(\mathbf{u})\mathbf{n}) \cdot \mathbf{v} \, ds \\
= \; & \int_{\Gamma_S} \left[(\mathbf{n}^t \sigma(\mathbf{u})\mathbf{n})\mathbf{n} + (\tau^t \sigma(\mathbf{u})\mathbf{n})\tau \right] \cdot \left[(\mathbf{n}^t \mathbf{v})\mathbf{n} + (\tau^t \mathbf{v})\tau \right] \, ds \\
= \; & \int_{\Gamma_S} (\mathbf{n}^t \sigma(\mathbf{u})\mathbf{n})(\mathbf{n}^t \mathbf{v}) \, ds + \int_{\Gamma_S} (\tau^t \sigma(\mathbf{u})\mathbf{n})(\tau^t \mathbf{v}) \, ds \\
= \; & \int_{\Gamma_S} p_n(\mathbf{n}^t \mathbf{v}) \, ds + \int_{\Gamma_S} p_\tau(\tau^t \mathbf{v}) \, ds.
\end{aligned}
$$

Stress boundary condition in normal/tangential directions

- Then the weak formulation is to find $\mathbf{u} \in H^1(\Omega) \times H^1(\Omega)$ such that

$$\int_\Omega \sigma(\mathbf{u}) : \nabla \mathbf{v} \; dx_1 dx_2$$
$$= \int_\Omega \mathbf{f} \cdot \mathbf{v} \; dx_1 dx_2 + \int_{\Gamma_S} p_n(\mathbf{n}^t \mathbf{v}) \; ds + \int_{\Gamma_S} p_\tau(\tau^t \mathbf{v}) \; ds.$$

for any $\mathbf{v} \in H^1_{0D}(\Omega) \times H^1_{0D}(\Omega)$.

Stress boundary condition in normal/tangential directions

- Then the Galerkin formulation is to find $\mathbf{u}_h \in U_h \times U_h$ such that

$$\int_\Omega \sigma(\mathbf{u}_h) : \nabla \mathbf{v}_h \, dx_1 dx_2$$
$$= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \, dx_1 dx_2 + \int_{\Gamma_S} p_n(\mathbf{n}^t \mathbf{v}_h) \, ds + \int_{\Gamma_S} p_\tau(\tau^t \mathbf{v}_h) \, ds.$$

for any $\mathbf{v}_h \in U_{h0} \times U_{h0}$.

- For an easier implementation, we use the following Galerkin formulation (without considering the Dirichlet boundary condition, which will be handled later): find $\mathbf{u}_h \in U_h \times U_h$ such that

$$\int_\Omega \sigma(\mathbf{u}_h) : \nabla \mathbf{v}_h \, dx_1 dx_2$$
$$= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \, dx_1 dx_2 + \int_{\Gamma_S} p_n(\mathbf{n}^t \mathbf{v}_h) \, ds + \int_{\Gamma_S} p_\tau(\tau^t \mathbf{v}_h) \, ds.$$

for any $\mathbf{v}_h \in U_h \times U_h$.

Stress boundary condition in normal/tangential directions

- Since $u_{1h},\ u_{2h} \in U_h = span\{\phi_j\}_{j=1}^{N_b}$, then

$$u_{1h} = \sum_{j=1}^{N_b} u_{1j}\phi_j, \quad u_{2h} = \sum_{j=1}^{N_b} u_{2j}\phi_j$$

  for some coefficients u_{1j} and u_{2j} $\ (j = 1, \cdots, N_b)$.

- For the test function, we choose $\mathbf{v}_h = (\phi_i, 0)^t$ $(i = 1, \cdots, N_b)$ and $\mathbf{v}_h = (0, \phi_i)^t$ $(i = 1, \cdots, N_b)$.

Stress boundary condition in normal/tangential directions

- Then by the same procedure to derive the matrix formulation before, we obtain the following linear system

$$
\begin{aligned}
&\sum_{j=1}^{N_b} u_{1j}\left(\int_\Omega \lambda \frac{\partial \phi_j}{\partial x_1}\frac{\partial \phi_i}{\partial x_1} dx_1 dx_2 + 2\int_\Omega \mu \frac{\partial \phi_j}{\partial x_1}\frac{\partial \phi_i}{\partial x_1} dx_1 dx_2 + \int_\Omega \mu \frac{\partial \phi_j}{\partial x_2}\frac{\partial \phi_i}{\partial x_2} dx_1 dx_2\right)\\
&+\sum_{j=1}^{N_b} u_{2j}\left(\int_\Omega \lambda \frac{\partial \phi_j}{\partial x_2}\frac{\partial \phi_i}{\partial x_1} dx_1 dx_2 + \int_\Omega \mu \frac{\partial \phi_j}{\partial x_1}\frac{\partial \phi_i}{\partial x_2} dx_1 dx_2\right)\\
&= \int_\Omega f_1 \phi_i dx_1 dx_2 + \int_{\Gamma_S} p_n \phi_i n_1 \ ds + \int_{\Gamma_S} p_\tau \phi_i \tau_1 \ ds\\
&\sum_{j=1}^{N_b} u_{1j}\left(\int_\Omega \lambda \frac{\partial \phi_j}{\partial x_1}\frac{\partial \phi_i}{\partial x_2} dx_1 dx_2 + \int_\Omega \mu \frac{\partial \phi_j}{\partial x_2}\frac{\partial \phi_i}{\partial x_1} dx_1 dx_2\right)\\
&+\sum_{j=1}^{N_b} u_{2j}\Big(\int_\Omega \lambda \frac{\partial \phi_j}{\partial x_2}\frac{\partial \phi_i}{\partial x_2} dx_1 dx_2 + 2\int_\Omega \mu \frac{\partial \phi_j}{\partial x_2}\frac{\partial \phi_i}{\partial x_2} dx_1 dx_2\\
&+\int_\Omega \mu \frac{\partial \phi_j}{\partial x_1}\frac{\partial \phi_i}{\partial x_1} dx_1 dx_2\Big)\\
&= \int_\Omega f_2 \phi_i dx_1 dx_2 + \int_{\Gamma_S} p_n \phi_i n_2 \ ds + \int_{\Gamma_S} p_\tau \phi_i \tau_2 \ ds.
\end{aligned}
$$

Stress boundary condition in normal/tangential directions

- Recall

$$A_1 = \left[\int_\Omega \lambda \frac{\partial \phi_j}{\partial x_1}\frac{\partial \phi_i}{\partial x_1} dx_1 dx_2\right]_{i,j=1}^{N_b}, A_2 = \left[\int_\Omega \mu \frac{\partial \phi_j}{\partial x_1}\frac{\partial \phi_i}{\partial x_1} dx_1 dx_2\right]_{i,j=1}^{N_b}$$

$$A_3 = \left[\int_\Omega \mu \frac{\partial \phi_j}{\partial x_2}\frac{\partial \phi_i}{\partial x_2} dx_1 dx_2\right]_{i,j=1}^{N_b}, A_4 = \left[\int_\Omega \lambda \frac{\partial \phi_j}{\partial x_2}\frac{\partial \phi_i}{\partial x_1} dx_1 dx_2\right]_{i,j=1}^{N_b}$$

$$A_5 = \left[\int_\Omega \mu \frac{\partial \phi_j}{\partial x_1}\frac{\partial \phi_i}{\partial x_2} dx_1 dx_2\right]_{i,j=1}^{N_b}, A_6 = \left[\int_\Omega \lambda \frac{\partial \phi_j}{\partial x_1}\frac{\partial \phi_i}{\partial x_2} dx_1 dx_2\right]_{i,j=1}^{N_b}$$

$$A_7 = \left[\int_\Omega \mu \frac{\partial \phi_j}{\partial x_2}\frac{\partial \phi_i}{\partial x_1} dx_1 dx_2\right]_{i,j=1}^{N_b}, A_8 = \left[\int_\Omega \lambda \frac{\partial \phi_j}{\partial x_2}\frac{\partial \phi_i}{\partial x_2} dx_1 dx_2\right]_{i,j=1}^{N_b}$$

and

$$A = \begin{pmatrix} A_1 + 2A_2 + A_3 & A_4 + A_5 \\ A_6 + A_7 & A_8 + 2A_3 + A_2 \end{pmatrix}$$

Stress boundary condition in normal/tangential directions

- Recall

$$\vec{b} = \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \end{pmatrix}$$

where

$$\vec{b}_1 = \left[\int_\Omega f_1 \phi_i dx_1 dx_2 \right]_{i=1}^{N_b}, \quad \vec{b}_2 = \left[\int_\Omega f_2 \phi_i dx_1 dx_2 \right]_{i=1}^{N_b}.$$

- Recall

$$\vec{X} = \begin{pmatrix} \vec{X}_1 \\ \vec{X}_2 \end{pmatrix}$$

where

$$\vec{X}_1 = [u_{1j}]_{j=1}^{N_b}, \quad \vec{X}_2 = [u_{2j}]_{j=1}^{N_b}.$$

Stress boundary condition in normal/tangential directions

- Define the additional vector from the stress boundary condition:

$$\vec{v} = \begin{pmatrix} \vec{v}_1 + \vec{v}_2 \\ \vec{v}_3 + \vec{v}_4 \end{pmatrix}$$

where

$$\vec{v}_1 = \left[\int_{\Gamma_S} p_n \phi_i n_1 \; ds\right]_{i=1}^{N_b}, \quad \vec{v}_2 = \left[\int_{\Gamma_S} p_\tau \phi_i \tau_1 \; ds\right]_{i=1}^{N_b},$$
$$\vec{v}_3 = \left[\int_{\Gamma_S} p_n \phi_i n_2 \; ds\right]_{i=1}^{N_b}, \quad \vec{v}_4 = \left[\int_{\Gamma_S} p_\tau \phi_i \tau_2 \; ds\right]_{i=1}^{N_b}.$$

- Define the new vector $\widetilde{\vec{b}} = \vec{b} + \vec{v}$.
- Then we obtain the linear algebraic system

$$A\vec{X} = \widetilde{\vec{b}}.$$

Stress boundary condition in normal/tangential directions

- Since each of $\vec{v}_i$ $(i = 1, 2, 3, 4)$ is similar to the $\vec{v}$ for the Neumann condition in Chapter 3, we can borrow the code of Neumman condition in Chapter 3 for $\vec{v}_i$ $(i = 1, 2, 3, 4)$.
- The major difference between $\vec{v}_i$ $(i = 1, 2, 3, 4)$ here and the $\vec{v}$ for the Neumann condition in Chapter 3 is that here we need to provide the unit normal/tangential vectors. That is, we need to provide $\mathbf{n} = (n_1, n_2)^t$ and $\tau = (\tau_1, \tau_2)^t$, in the information matrix $boundaryedges$.

Stress boundary condition in normal/tangential directions

Based on Algorithm VI in Chapter 3, we obtain Algorithm VI-3:

- Initialize the vector: $v = sparse(2N_b, 1)$;
- Compute the integrals and assemble them into v:

$FOR\ k = 1, \cdots, nbe$:

$IF\ boundaryedges(1, k)$ shows stress boundary in normal/tangential directions, $THEN$

$n_k = boundaryedges(2, k)$;

$FOR\ \beta = 1, \cdots, N_{lb}$:

Compute

$$r = \int_{e_k} p_n \frac{\partial^{a+b}\psi_{n_k\beta}}{\partial x_1^a \partial x_2^b} n_1\ ds + \int_{e_k} p_\tau \frac{\partial^{a+b}\psi_{n_k\beta}}{\partial x_1^a \partial x_2^b} \tau_1\ ds;$$

$$v(T_b(\beta, n_k), 1) = v(T_b(\beta, n_k), 1) + r;$$

Compute

$$r = \int_{e_k} p_n \frac{\partial^{a+b}\psi_{n_k\beta}}{\partial x_1^a \partial x_2^b} n_2\ ds + \int_{e_k} p_\tau \frac{\partial^{a+b}\psi_{n_k\beta}}{\partial x_1^a \partial x_2^b} \tau_2\ ds;$$

$$v(N_b + T_b(\beta, n_k), 1) = v(N_b + T_b(\beta, n_k), 1) + r;$$

END

$ENDIF$

END

Robin boundary conditions in normal/tangential directions

- Consider

$$
\begin{aligned}
&-\nabla \cdot \sigma(\mathbf{u}) = \mathbf{f} \qquad in \quad \Omega, \\
&\mathbf{n}^t\sigma(\mathbf{u})\mathbf{n} + r\mathbf{n}^t\mathbf{u} = q_n, \;\; \tau^t\sigma(\mathbf{u})\mathbf{n} + r\tau^t\mathbf{u} = q_\tau \;\; \text{on } \Gamma_R \subseteq \partial\Omega, \\
&\mathbf{u} = \mathbf{g} \;\; \text{on } \Gamma_D = \partial\Omega/\Gamma_R.
\end{aligned}
$$

- Recall

$$
\int_\Omega \sigma(\mathbf{u}) : \nabla\mathbf{v} \; dx_1dx_2 - \int_{\partial\Omega} (\sigma(\mathbf{u})\mathbf{n}) \cdot \mathbf{v} \; ds = \int_\Omega \mathbf{f} \cdot \mathbf{v} \; dx_1dx_2.
$$

- Since the solution on $\Gamma_D = \partial\Omega/\Gamma_R$ is given by $\mathbf{u} = \mathbf{g}$, then we can choose the test function $\mathbf{v}(x_1, x_2)$ such that $\mathbf{v} = 0$ on $\partial\Omega/\Gamma_R$.

Robin boundary condition in normal/tangential directions

- Using the above conditions, orthogonal decomposition of a vector, and the definition of unit normal/tangential vector, we obtain

$$
\begin{aligned}
& \int_{\partial\Omega} (\sigma(\mathbf{u})\mathbf{n}) \cdot \mathbf{v} \; ds \\
= \; & \int_{\Gamma_R} (\sigma(\mathbf{u})\mathbf{n}) \cdot \mathbf{v} \; ds \\
= \; & \int_{\Gamma_R} \left[(\mathbf{n}^t \sigma(\mathbf{u})\mathbf{n})\mathbf{n} + (\tau^t \sigma(\mathbf{u})\mathbf{n})\tau\right] \cdot \left[(\mathbf{n}^t \mathbf{v})\mathbf{n} + (\tau^t \mathbf{v})\tau\right] \; ds \\
= \; & \int_{\Gamma_S} (\mathbf{n}^t \sigma(\mathbf{u})\mathbf{n})(\mathbf{n}^t \mathbf{v}) \; ds + \int_{\Gamma_S} (\tau^t \sigma(\mathbf{u})\mathbf{n})(\tau^t \mathbf{v}) \; ds \\
= \; & \left[\int_{\Gamma_R} q_n (\mathbf{n}^t \mathbf{v}) \; ds + \int_{\Gamma_R} q_\tau (\tau^t \mathbf{v}) \; ds\right] \\
& - \left[\int_{\Gamma_R} (r\mathbf{n}^t \mathbf{u})(\mathbf{n}^t \mathbf{v}) \; ds + \int_{\Gamma_R} (r\tau^t \mathbf{u})(\tau^t \mathbf{v}) \; ds\right],
\end{aligned}
$$

Robin boundary condition in normal/tangential directions

- Then the weak formulation is to find $\mathbf{u} \in H^1(\Omega) \times H^1(\Omega)$ such that

$$\int_\Omega \sigma(\mathbf{u}) : \nabla \mathbf{v} \; dx_1 dx_2 + \int_{\Gamma_R} (r\mathbf{n}^t\mathbf{u})(\mathbf{n}^t\mathbf{v}) \; ds + \int_{\Gamma_R} (r\tau^t\mathbf{u})(\tau^t\mathbf{v}) \; ds$$
$$= \int_\Omega \mathbf{f} \cdot \mathbf{v} \; dx_1 dx_2 + \int_{\Gamma_R} q_n(\mathbf{n}^t\mathbf{v}) \; ds + \int_{\Gamma_R} q_\tau(\tau^t\mathbf{v}) \; ds.$$

for any $\mathbf{v} \in H^1_{0D}(\Omega) \times H^1_{0D}(\Omega)$.

Robin boundary condition in normal/tangential directions

- Then the Galerkin formulation is to find $\mathbf{u}_h \in U_h \times U_h$ such that

$$\int_\Omega \sigma(\mathbf{u}_h) : \nabla \mathbf{v}_h \, dx_1 dx_2 + \int_{\Gamma_R} (r\mathbf{n}^t \mathbf{u}_h)(\mathbf{n}^t \mathbf{v}_h) \, ds + \int_{\Gamma_R} (r\tau^t \mathbf{u}_h)(\tau^t \mathbf{v}_h) \, ds$$

$$= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \, dx_1 dx_2 + \int_{\Gamma_R} q_n (\mathbf{n}^t \mathbf{v}_h) \, ds + \int_{\Gamma_R} q_\tau (\tau^t \mathbf{v}_h) \, ds.$$

for any $\mathbf{v}_h \in U_{h0} \times U_{h0}$.

- For an easier implementation, we use the following Galerkin formulation (without considering the Dirichlet boundary condition, which will be handled later): find $\mathbf{u}_h \in U_h \times U_h$ such that

$$\int_\Omega \sigma(\mathbf{u}_h) : \nabla \mathbf{v}_h \, dx_1 dx_2 + \int_{\Gamma_R} (r\mathbf{n}^t \mathbf{u}_h)(\mathbf{n}^t \mathbf{v}_h) \, ds + \int_{\Gamma_R} (r\tau^t \mathbf{u}_h)(\tau^t \mathbf{v}_h) \, ds$$

$$= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \, dx_1 dx_2 + \int_{\Gamma_R} q_n (\mathbf{n}^t \mathbf{v}_h) \, ds + \int_{\Gamma_R} q_\tau (\tau^t \mathbf{v}_h) \, ds.$$

for any $\mathbf{v}_h \in U_h \times U_h$.

Robin boundary condition in normal/tangential directions

- Since $u_{1h},\ u_{2h} \in U_h = span\{\phi_j\}_{j=1}^{N_b}$, then

$$u_{1h} = \sum_{j=1}^{N_b} u_{1j}\phi_j, \quad u_{2h} = \sum_{j=1}^{N_b} u_{2j}\phi_j$$

  for some coefficients u_{1j} and u_{2j} $\ (j = 1, \cdots, N_b)$.

- For the test function, we choose $\mathbf{v}_h = (\phi_i, 0)^t$ $(i = 1, \cdots, N_b)$ and $\mathbf{v}_h = (0, \phi_i)^t$ $(i = 1, \cdots, N_b)$.

Robin boundary condition in normal/tangential directions

- Then by the same procedure to derive the matrix formulation before, we obtain the following linear system:

$$
\begin{aligned}
&\sum_{j=1}^{N_b} u_{1j}\Big(\int_\Omega (\lambda+2\mu)\frac{\partial \phi_j}{\partial x_1}\frac{\partial \phi_i}{\partial x_1}dx_1dx_2+\int_\Omega \mu\frac{\partial \phi_j}{\partial x_2}\frac{\partial \phi_i}{\partial x_2}dx_1dx_2\\
&+\int_{\Gamma_R}(rn_1\phi_j)(\phi_i n_1)\ ds+\int_{\Gamma_R}(r\tau_1\phi_j)(\phi_i\tau_1)\ ds\Big)\\
&+\sum_{j=1}^{N_b} u_{2j}\Big(\int_\Omega \lambda\frac{\partial \phi_j}{\partial x_2}\frac{\partial \phi_i}{\partial x_1}dx_1dx_2+\int_\Omega \mu\frac{\partial \phi_j}{\partial x_1}\frac{\partial \phi_i}{\partial x_2}dx_1dx_2\\
&+\int_{\Gamma_R}(rn_2\phi_j)(\phi_i n_1)\ ds+\int_{\Gamma_R}(r\tau_2\phi_j)(\phi_i\tau_1)\ ds\Big)\\
&=\int_\Omega f_1\phi_i dx_1dx_2+\int_{\Gamma_R} q_n\phi_i n_1\ ds+\int_{\Gamma_R} q_\tau\phi_i\tau_1\ ds,
\end{aligned}
$$

Robin boundary condition in normal/tangential directions

- and

$$
\begin{aligned}
&\sum_{j=1}^{N_b} u_{1j}\Big(\int_\Omega \lambda \frac{\partial \phi_j}{\partial x_1}\frac{\partial \phi_i}{\partial x_2} dx_1dx_2 + \int_\Omega \mu \frac{\partial \phi_j}{\partial x_2}\frac{\partial \phi_i}{\partial x_1} dx_1dx_2 \\
&+ \int_{\Gamma_R} (rn_1\phi_j)(\phi_i n_2)\ ds + \int_{\Gamma_R} (r\tau_1\phi_j)(\phi_i\tau_2)\ ds\Big) \\
&+ \sum_{j=1}^{N_b} u_{2j}\Big(\int_\Omega (\lambda+2\mu) \frac{\partial \phi_j}{\partial x_2}\frac{\partial \phi_i}{\partial x_2} dx_1dx_2 + \int_\Omega \mu \frac{\partial \phi_j}{\partial x_1}\frac{\partial \phi_i}{\partial x_1} dx_1dx_2 \\
&+ \int_{\Gamma_R} (rn_2\phi_j)(\phi_i n_2)\ ds + \int_{\Gamma_R} (r\tau_2\phi_j)(\phi_i\tau_2)\ ds\Big) \\
&= \int_\Omega f_2\phi_i dx_1dx_2 + \int_{\Gamma_R} q_n\phi_i n_2\ ds + \int_{\Gamma_R} q_\tau\phi_i\tau_2\ ds.
\end{aligned}
$$

Robin boundary condition in normal/tangential directions

- Matrix formulation? Pesudo code? (Part of a project for you)
- Similar to the previous ones for Robin condition, we need to add eight sub-matrices and four sub-vectors into the block linear system.
- The major difference is that here we need to provide the unit normal/tangential vectors. That is, we need to provide $\mathbf{n} = (n_1, n_2)^t$ and $\tau = (\tau_1, \tau_2)^t$, in the information matrix $boundaryedges$.

Dirichlet/stress/Robin mixed boundary condition in normal/tangential directions

- Consider

$$
\begin{aligned}
&-\nabla \cdot \sigma(\mathbf{u}) = \mathbf{f} \qquad in \quad \Omega, \\
&\mathbf{n}^t\sigma(\mathbf{u})\mathbf{n} = p_n, \;\; \tau^t\sigma(\mathbf{u})\mathbf{n} = p_\tau \;\; \text{on } \Gamma_S \subset \partial\Omega, \\
&\mathbf{n}^t\sigma(\mathbf{u})\mathbf{n} + r\mathbf{n}^t\mathbf{u} = q_n, \;\; \tau^t\sigma(\mathbf{u})\mathbf{n} + r\tau^t\mathbf{u} = q_\tau \;\; \text{on } \Gamma_R \subseteq \partial\Omega, \\
&\mathbf{u} = \mathbf{g} \;\; \text{on } \Gamma_D = \partial\Omega/(\Gamma_S \cup \Gamma_R).
\end{aligned}
$$

- Recall

$$
\int_\Omega \sigma(\mathbf{u}) : \nabla\mathbf{v} \; dx_1dx_2 - \int_{\partial\Omega} (\sigma(\mathbf{u})\mathbf{n}) \cdot \mathbf{v} \; ds = \int_\Omega \mathbf{f} \cdot \mathbf{v} \; dx_1dx_2.
$$

- Since the solution on $\Gamma_D = \partial\Omega/(\Gamma_S \cup \Gamma_R)$ is given by $\mathbf{u} = \mathbf{g}$, then we can choose the test function $\mathbf{v}(x_1, x_2)$ such that $\mathbf{v} = 0$ on $\partial\Omega/(\Gamma_S \cup \Gamma_R)$.

Dirichlet/stress/Robin mixed boundary condition in normal/tangential directions

- Combining the above derivation for stress and Robin boundary conditions in normal/tangential directions, we obtain

$$
\begin{aligned}
&\int_{\partial\Omega} (\sigma(\mathbf{u})\mathbf{n}) \cdot \mathbf{v}\ ds \\
= \ & \int_{\Gamma_S} (\sigma(\mathbf{u})\mathbf{n}) \cdot \mathbf{v}\ ds + \int_{\Gamma_R} (\sigma(\mathbf{u})\mathbf{n}) \cdot \mathbf{v}\ ds + \int_{\partial\Omega/(\Gamma_S \cup \Gamma_R)} (\sigma(\mathbf{u})\mathbf{n}) \cdot \mathbf{v}\ ds \\
= \ & \left[\int_{\Gamma_S} p_n(\mathbf{n}^t\mathbf{v})\ ds + \int_{\Gamma_S} p_\tau(\tau^t\mathbf{v})\ ds\right] \\
& + \left[\int_{\Gamma_R} q_n(\mathbf{n}^t\mathbf{v})\ ds + \int_{\Gamma_R} q_\tau(\tau^t\mathbf{v})\ ds\right] \\
& - \left[\int_{\Gamma_R} (r\mathbf{n}^t\mathbf{u})(\mathbf{n}^t\mathbf{v})\ ds + \int_{\Gamma_R} (r\tau^t\mathbf{u})(\tau^t\mathbf{v})\ ds\right],
\end{aligned}
$$

Dirichlet/stress/Robin mixed boundary condition in normal/tangential directions

- The weak formulation is to find $\mathbf{u} \in H^1(\Omega) \times H^1(\Omega)$ such that

$$
\begin{aligned}
& \int_\Omega \sigma(\mathbf{u}) : \nabla \mathbf{v} \, dx_1 dx_2 + \int_{\Gamma_R} (r\mathbf{n}^t\mathbf{u})(\mathbf{n}^t\mathbf{v}) \, ds + \int_{\Gamma_R} (r\tau^t\mathbf{u})(\tau^t\mathbf{v}) \, ds \\
= \; & \int_\Omega \mathbf{f} \cdot \mathbf{v} \, dx_1 dx_2 + \int_{\Gamma_R} q_n(\mathbf{n}^t\mathbf{v}) \, ds + \int_{\Gamma_R} q_\tau(\tau^t\mathbf{v}) \, ds \\
& + \int_{\Gamma_S} p_n(\mathbf{n}^t\mathbf{v}) \, ds + \int_{\Gamma_S} p_\tau(\tau^t\mathbf{v}) \, ds.
\end{aligned}
$$

  for any $\mathbf{v} \in H^1_{0D}(\Omega) \times H^1_{0D}(\Omega)$.
- Code?
- Combine all of the subroutines for Dirichlet/stress/Robin boundary conditions.