# Introduction and Basic Implementation for Finite Element Methods

## Chapter 9: Finite elements for 2D unsteady Navier-Stokes equations

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# Outline

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# Target problem

- Consider the 2D unsteady unsteady Navier-Stokes equation equation

$$
\begin{aligned}
&\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla \cdot \mathbb{T}(\mathbf{u}, p) = \mathbf{f}, \quad \text{in } \Omega \times [0, T], \\
&\nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega \times [0, T], \\
&\mathbf{u} = \mathbf{g}, \quad \text{on } \partial\Omega \times [0, T], \\
&\mathbf{u} = \mathbf{u}_0, \;\; p = p_0, \quad \text{at } t = 0 \text{ and in } \Omega.
\end{aligned}
$$

where $\Omega$ is a 2D domain, $[0, T]$ is the time interval, $\mathbf{f}(x, y, t)$ is a given function on $\Omega \times [0, T]$, $\mathbf{g}(x, y, t)$ is a given function on $\partial\Omega \times [0, T]$, $\mathbf{u}_0(x, y)$ and $p_0(x, y)$ are given functions on $\Omega$ at $t = 0$, $\mathbf{u}(x, y, t)$ and $p(x, y, t)$ are the unknown functions, and

$$
\begin{aligned}
&\mathbf{u}(x, y, t) = (u_1, \; u_2)^t, \quad \mathbf{f}(x, y, t) = (f_1, \; f_2)^t, \\
&\mathbf{g}(x, y, t) = (g_1, \; g_2)^t, \quad \mathbf{u}_0(x, y) = (u_{10}, \; u_{20})^t.
\end{aligned}
$$

# Target problem

- The nonlinear advection is defined as

$$(\mathbf{u} \cdot \nabla)\mathbf{u} = \begin{pmatrix} u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} \\ u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} \end{pmatrix}.$$

- The stress tensor $\mathbb{T}(\mathbf{u}, p)$ is defined as

$$\mathbb{T}(\mathbf{u}, p) = 2\nu\mathbb{D}(\mathbf{u}) - p\mathbb{I}$$

where $\nu$ is the viscosity and the deformation tensor

$$\mathbb{D}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^t).$$

# Target problem

- In more details, the deformation tensor can be written as

$$\mathbb{D}(\mathbf{u}) = \begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) & \frac{\partial u_2}{\partial y} \end{pmatrix}.$$

- Hence the stress tensor can be written as

$$\mathbb{T}(\mathbf{u}, p) = \begin{pmatrix} 2\nu\frac{\partial u_1}{\partial x} - p & \nu\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) \\ \nu\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) & 2\nu\frac{\partial u_2}{\partial y} - p \end{pmatrix}.$$

# Weak formulation

- First, take the inner product with a vector function $\mathbf{v}(x, y) = (v_1, v_2)^t$ on both sides of the unsteady Navier-Stokes equation:

$$
\begin{aligned}
& \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla \cdot \mathbb{T}(\mathbf{u}, p) = \mathbf{f} \quad \text{in } \Omega \\
\Rightarrow \quad & \mathbf{u}_t \cdot \mathbf{v} + (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v} - \nabla \cdot \mathbb{T}(\mathbf{u}, p) \cdot \mathbf{v} = \mathbf{f} \cdot \mathbf{v} \quad \text{in } \Omega \\
\Rightarrow \quad & \int_\Omega \mathbf{u}_t \cdot \mathbf{v} \; dxdy + \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v} \; dxdy \\
& - \int_\Omega (\nabla \cdot \mathbb{T}(\mathbf{u}, p)) \cdot \mathbf{v} \; dxdy = \int_\Omega \mathbf{f} \cdot \mathbf{v} \; dxdy
\end{aligned}
$$

# Weak formulation

- Second, multiply the divergence free equation by a function $q(x, y)$:

$$\nabla \cdot \mathbf{u} = 0 \quad \Rightarrow \quad (\nabla \cdot \mathbf{u})q = 0$$
$$\Rightarrow \quad \int_{\Omega} (\nabla \cdot \mathbf{u})q \; dxdy = 0.$$

- $\mathbf{u}(x, y, t)$ and $p(x, y, t)$ are called trail functions and $\mathbf{v}(x, y)$ and $q(x, y)$ are called test functions.

# Weak formulation

- Using integration by parts in multi-dimension:

$$\int_\Omega (\nabla \cdot \mathbb{T}) \cdot \mathbf{v} \; dxdy = \int_{\partial\Omega} (\mathbb{T}\mathbf{n}) \cdot \mathbf{v} \; ds - \int_\Omega \mathbb{T} : \nabla \mathbf{v} \; dxdy,$$

where $\mathbf{n} = (n_1, \; n_2)^t$ is the unit outer normal vector of $\partial\Omega$, we obtain

$$\int_\Omega \mathbf{u}_t \cdot \mathbf{v} \; dxdy + \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v} \; dxdy + \int_\Omega \mathbb{T}(\mathbf{u}, p) : \nabla \mathbf{v} \; dxdy$$
$$- \int_{\partial\Omega} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \; ds = \int_\Omega \mathbf{f} \cdot \mathbf{v} \; dxdy.$$

Here,

$$\begin{aligned} A : B &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ &= a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}. \end{aligned}$$

# Weak formulation

- Using the above definition for $A : B$, it is not difficult to verify (an independent study project topic) that

$$\begin{aligned} \mathbb{T}(\mathbf{u}, p) : \nabla \mathbf{v} &= (2\nu\mathbb{D}(\mathbf{u}) - p\mathbb{I}) : \nabla \mathbf{v} \\ &= 2\nu\mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) - p(\nabla \cdot \mathbf{v}). \end{aligned}$$

- Hence we obtain

$$\int_\Omega \mathbf{u}_t \cdot \mathbf{v}\ dxdy + \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v}\ dxdy + \int_\Omega 2\nu\mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v})\ dxdy$$

$$- \int_\Omega p(\nabla \cdot \mathbf{v})\ dxdy - \int_{\partial\Omega} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v}\ ds = \int_\Omega \mathbf{f} \cdot \mathbf{v}\ dxdy,$$

$$- \int_\Omega (\nabla \cdot \mathbf{u})q\ dxdy = 0.$$

Here we multiply the second equation by $-1$ in order to keep the matrix formulation symmetric later.

# Weak formulation

- Since the solution on the domain boundary $\partial\Omega$ are given by $\mathbf{u}(x,y,t) = \mathbf{g}(x,y,t)$, then we can choose the test function $\mathbf{v}(x,y)$ such that $\mathbf{v} = 0$ on $\partial\Omega$.
- Hence

$$\int_\Omega \mathbf{u}_t \cdot \mathbf{v}\ dxdy + \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\ dxdy + \int_\Omega 2\nu\mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v})\ dxdy$$

$$-\int_\Omega p(\nabla\cdot\mathbf{v})\ dxdy = \int_\Omega \mathbf{f}\cdot\mathbf{v}\ dxdy,$$

$$-\int_\Omega (\nabla\cdot\mathbf{u})q\ dxdy = 0.$$

- Define $[H^1(\Omega)]^2 = H^1(\Omega)\times H^1(\Omega)$ and

$$\begin{aligned} H^1(0,T;[H^1(\Omega)]^2) &= \{\mathbf{v}(\cdot,t),\ \frac{\partial\mathbf{v}}{\partial t}(\cdot,t)\in[H^1(\Omega)]^2,\ \forall t\in[0,T]\}, \\ L^2(0,T;L^2(\Omega)) &= \{q(\cdot,t)\in L^2(\Omega),\ \forall t\in[0,T]\}. \end{aligned}$$

# Weak formulation

- Weak formulation in the vector format: find $\mathbf{u} \in H^1(0, T; [H^1(\Omega)]^2)$ and $p \in L^2(0, T; L^2(\Omega))$ such that

$$\begin{aligned}
&\int_\Omega \mathbf{u}_t \cdot \mathbf{v}\ dxdy + \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v}\ dxdy \\
&+ \int_\Omega 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v})\ dxdy - \int_\Omega p(\nabla \cdot \mathbf{v})\ dxdy \\
&= \int_\Omega \mathbf{f} \cdot \mathbf{v}\ dxdy, \\
&- \int_\Omega (\nabla \cdot \mathbf{u}) q\ dxdy = 0,
\end{aligned}$$

for any $\mathbf{v} \in [H_0^1(\Omega)]^2$ and $q \in L^2(\Omega)$.

# Weak formulation

- Define

$$c(\mathbf{w},\mathbf{u},\mathbf{v}) = \int_\Omega (\mathbf{w}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\ dxdy,$$

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega 2\nu\mathbb{D}(\mathbf{u}):\mathbb{D}(\mathbf{v})\ dxdy,$$

$$b(\mathbf{u},q) = -\int_\Omega (\nabla\cdot\mathbf{u})q\ dxdy,$$

$$(\mathbf{f},\mathbf{v}) = \int_\Omega \mathbf{f}\cdot\mathbf{v}\ dxdy.$$

- Weak formulation: find $\mathbf{u}\in H^1(0,T;[H^1(\Omega)]^2)$ and $p\in L^2(0,T;L^2(\Omega))$ such that

$$\begin{aligned}(\mathbf{u}_t,\mathbf{v}) + c(\mathbf{u},\mathbf{u},\mathbf{v}) + a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) &= (\mathbf{f},\mathbf{v}),\\ b(\mathbf{u},q) &= 0,\end{aligned}$$

for any $\mathbf{v}\in[H_0^1(\Omega)]^2$ and $q\in L^2(\Omega)$.

# Weak formulation

- In more details,

$$
\begin{aligned}
&\mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \\
= & \begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) & \frac{\partial u_2}{\partial y} \end{pmatrix} \\
& : \begin{pmatrix} \frac{\partial v_1}{\partial x} & \frac{1}{2}\left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x}\right) & \frac{\partial v_2}{\partial y} \end{pmatrix} \\
= & \frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial x} + \frac{1}{4}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right)\left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x}\right) \\
& + \frac{1}{4}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right)\left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x}\right) + \frac{\partial u_2}{\partial y}\frac{\partial v_2}{\partial y}.
\end{aligned}
$$

# Weak formulation

- Hence

$$
\begin{aligned}
& \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \\
= \; & \frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial x} + \frac{\partial u_2}{\partial y}\frac{\partial v_2}{\partial y} + \frac{1}{2}\frac{\partial u_1}{\partial y}\frac{\partial v_1}{\partial y} \\
& + \frac{1}{2}\frac{\partial u_1}{\partial y}\frac{\partial v_2}{\partial x} + \frac{1}{2}\frac{\partial u_2}{\partial x}\frac{\partial v_1}{\partial y} + \frac{1}{2}\frac{\partial u_2}{\partial x}\frac{\partial v_2}{\partial x}.
\end{aligned}
$$

- Then

$$
\begin{aligned}
& \int_\Omega 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \; dxdy \\
= \; & \int_\Omega \nu \Big( 2\frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial x} + 2\frac{\partial u_2}{\partial y}\frac{\partial v_2}{\partial y} + \frac{\partial u_1}{\partial y}\frac{\partial v_1}{\partial y} \\
& + \frac{\partial u_1}{\partial y}\frac{\partial v_2}{\partial x} + \frac{\partial u_2}{\partial x}\frac{\partial v_1}{\partial y} + \frac{\partial u_2}{\partial x}\frac{\partial v_2}{\partial x} \Big) \; dxdy.
\end{aligned}
$$

# Weak formulation

- We also have

$$\int_\Omega \mathbf{u}_t \cdot \mathbf{v}\ dxdy = \int_\Omega \frac{\partial u_1}{\partial t} v_1\ dxdy + \int_\Omega \frac{\partial u_2}{\partial t} v_2\ dxdy,$$

$$\int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v}\ dxdy$$

$$= \int_\Omega \left( u_1 \frac{\partial u_1}{\partial x} v_1 + u_2 \frac{\partial u_1}{\partial y} v_1 + u_1 \frac{\partial u_2}{\partial x} v_2 + u_2 \frac{\partial u_2}{\partial y} v_2 \right)\ dxdy,$$

$$\int_\Omega p(\nabla \cdot \mathbf{v})\ dxdy = \int_\Omega \left( p\frac{\partial v_1}{\partial x} + p\frac{\partial v_2}{\partial y} \right)\ dxdy,$$

$$\int_\Omega \mathbf{f} \cdot \mathbf{v}\ dxdy = \int_\Omega (f_1 v_1 + f_2 v_2)\ dxdy,$$

$$\int_\Omega (\nabla \cdot \mathbf{u}) q\ dxdy = \int_\Omega \left( \frac{\partial u_1}{\partial x} q + \frac{\partial u_2}{\partial y} q \right)\ dxdy.$$

# Weak formulation

- Weak formulation in the scalar format: find $u_1 \in H^1(\Omega)$, $u_2 \in H^1(\Omega)$, and $p \in L^2(\Omega)$ such that

$$\int_\Omega \frac{\partial u_1}{\partial t} v_1 \, dxdy + \int_\Omega \frac{\partial u_2}{\partial t} v_2 \, dxdy$$
$$+ \int_\Omega \left( u_1 \frac{\partial u_1}{\partial x} v_1 + u_2 \frac{\partial u_1}{\partial y} v_1 + u_1 \frac{\partial u_2}{\partial x} v_2 + u_2 \frac{\partial u_2}{\partial y} v_2 \right) \, dxdy$$
$$+ \int_\Omega \nu \Big( 2 \frac{\partial u_1}{\partial x} \frac{\partial v_1}{\partial x} + 2 \frac{\partial u_2}{\partial y} \frac{\partial v_2}{\partial y} + \frac{\partial u_1}{\partial y} \frac{\partial v_1}{\partial y}$$
$$+ \frac{\partial u_1}{\partial y} \frac{\partial v_2}{\partial x} + \frac{\partial u_2}{\partial x} \frac{\partial v_1}{\partial y} + \frac{\partial u_2}{\partial x} \frac{\partial v_2}{\partial x} \Big) \, dxdy$$
$$- \int_\Omega \left( p \frac{\partial v_1}{\partial x} + p \frac{\partial v_2}{\partial y} \right) \, dxdy = \int_\Omega (f_1 v_1 + f_2 v_2) \, dxdy.$$
$$- \int_\Omega \left( \frac{\partial u_1}{\partial x} q + \frac{\partial u_2}{\partial y} q \right) \, dxdy = 0.$$

for any $v_1 \in H_0^1(\Omega)$, $v_2 \in H_0^1(\Omega)$, and $q \in L^2(\Omega)$.

# Outline

# Galerkin formulation

- Consider a finite element space $U_h \subset H^1(\Omega)$ for the velocity and a finite element space $W_h \subset L^2(\Omega)$ for the pressure. Define $U_{h0}$ to be the space which consists of the functions of $U_h$ with value 0 on the Dirichlet boundary.

- Then the Galerkin formulation is to find $\mathbf{u}_h \in H^1(0, T; [U_h]^2)$ and $p \in L^2(0, T; W_h)$ such that

$$
\begin{aligned}
(\mathbf{u}_{h_t}, \mathbf{v}) + c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h), \\
b(\mathbf{u}_h, q_h) &= 0,
\end{aligned}
$$

for any $\mathbf{v}_h \in [U_{h0}]^2$ and $q_h \in W_h$.

# Galerkin formulation

- For an easier implementation, we use the following Galerkin formulation (without considering the Dirichlet boundary condition, which will be handled later): find $\mathbf{u}_h \in H^1(0, T; [U_h]^2)$ and $p \in L^2(0, T; W_h)$ such that

$$
\begin{aligned}
(\mathbf{u}_{h_t}, \mathbf{v}) + c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h), \\
b(\mathbf{u}_h, q_h) &= 0,
\end{aligned}
$$

for any $\mathbf{v}_h \in [U_h]^2$ and $q_h \in W_h$.

# Galerkin formulation

- In more details of the vector format, the Galerkin formulation (without considering the Dirichlet boundary condition, which will be handled later) is to find $\mathbf{u}_h \in H^1(0, T; [U_h]^2)$ and $p \in L^2(0, T; W_h)$ such that

$$\int_\Omega \mathbf{u}_{h_t} \cdot \mathbf{v}_h \; dxdy + \int_\Omega (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h \cdot \mathbf{v}_h \; dxdy$$
$$+ \int_\Omega 2\nu \mathbb{D}(\mathbf{u}_h) : \mathbb{D}(\mathbf{v}_h) \; dxdy - \int_\Omega p_h(\nabla \cdot \mathbf{v}_h) \; dxdy$$
$$= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \; dxdy,$$
$$- \int_\Omega (\nabla \cdot \mathbf{u}_h) q_h \; dxdy = 0,$$

for any $\mathbf{v}_h \in [U_h]^2$ and $q_h \in W_h$.

# Galerkin formulation

- In our numerical example, $U_h = span\{\phi_j\}_{j=1}^{N_b}$ and $W_h = span\{\psi_j\}_{j=1}^{N_{bp}}$ are chosen to be the finite element spaces with the quadratic global basis functions $\{\phi_j\}_{j=1}^{N_b}$ and linear global basis functions $\{\psi_j\}_{j=1}^{N_{bp}}$, which are defined in Chapter 2. They are called Taylor-Hood finite elements.
- Why do we choose the pairs of finite elements in this way?
- Stability of mixed finite elements: inf-sup condition.

$$\inf_{0 \neq q_h \in W_h} \sup_{0 \neq \mathbf{u}_h \in U_h \times U_h} \frac{b(\mathbf{u}_h, q_h)}{\|\nabla \mathbf{u}_h\|_0 \, \|q_h\|_0} > \beta,$$

where $\beta > 0$ is a constant independent of mesh size $h$.
- See other course materials and references for the theory and more examples of stable mixed finite elements for unsteady Navier-Stokes equation.

# Galerkin formulation

- In the scalar format, the Galerkin formulation is to find $u_{1h} \in H^1(0,T;U_h)$, $u_{2h} \in H^1(0,T;U_h)$, and $p_h \in L^2(0,T;W_h)$ such that

$$
\begin{aligned}
&\int_\Omega \frac{\partial u_{1h}}{\partial t} v_{1h}\, dxdy + \int_\Omega \frac{\partial u_{2h}}{\partial t} v_{2h}\, dxdy \\
&+ \int_\Omega \left( u_{1h}\frac{\partial u_{1h}}{\partial x} v_{1h} + u_{2h}\frac{\partial u_{1h}}{\partial y} v_{1h} + u_{1h}\frac{\partial u_{2h}}{\partial x} v_{2h} + u_{2h}\frac{\partial u_{2h}}{\partial y} v_{2h} \right)\, dxdy \\
&+ \int_\Omega \nu \Big( 2\frac{\partial u_{1h}}{\partial x}\frac{\partial v_{1h}}{\partial x} + 2\frac{\partial u_{2h}}{\partial y}\frac{\partial v_{2h}}{\partial y} + \frac{\partial u_{1h}}{\partial y}\frac{\partial v_{1h}}{\partial y} \\
&+ \frac{\partial u_{1h}}{\partial y}\frac{\partial v_{2h}}{\partial x} + \frac{\partial u_{2h}}{\partial x}\frac{\partial v_{1h}}{\partial y} + \frac{\partial u_{2h}}{\partial x}\frac{\partial v_{2h}}{\partial x} \Big)\, dxdy \\
&- \int_\Omega \left( p_h \frac{\partial v_{1h}}{\partial x} + p_h \frac{\partial v_{2h}}{\partial y} \right)\, dxdy = \int_\Omega (f_1 v_{1h} + f_2 v_{2h})\, dxdy. \\
&- \int_\Omega \left( \frac{\partial u_{1h}}{\partial x} q_h + \frac{\partial u_{2h}}{\partial y} q_h \right)\, dxdy = 0.
\end{aligned}
$$

for any $v_{1h} \in U_h$, $v_{2h} \in U_h$, and $q_h \in W_h$.

# Outline

# Full discretization

- Assume that we have a uniform partition of $[0, T]$ into $M_m$ elements with mesh size $\triangle t$.
- The mesh nodes are $t_m = m\triangle t,\ m = 0, 1, \cdots, M_m$.
- Let $\mathbf{u}_h^0$ and $p_h^0$ denote the given initial condition at $t_0$.
- Let $\mathbf{u}_h^m$ and $p_h^m$ denote the numerical solution at $t_m$.
- For a simple illustration, we consider the full discretization with backward Euler scheme (without considering the Dirichlet boundary condition, which will be handled later): for $m = 0, \cdots, M_m - 1$, find $\mathbf{u}_h^{m+1} \in [U_h]^2$ and $p_h^{m+1} \in W_h$ such that

$$
\begin{aligned}
&\left(\frac{\mathbf{u}_h^{m+1} - \mathbf{u}_h^m}{\triangle t}, \mathbf{v}\right) + c(\mathbf{u}_h^{m+1}, \mathbf{u}_h^{m+1}, \mathbf{v}_h) + a(\mathbf{u}_h^{m+1}, \mathbf{v}_h) \\
&+ b(\mathbf{v}_h, p_h^{m+1}) = (\mathbf{f}(t_{m+1}), \mathbf{v}_h), \\
&b(\mathbf{u}_h^{m+1}, q_h) = 0,
\end{aligned}
$$

# Full discretization

- That is, for $m = 0, \cdots, M_m - 1$, find $\mathbf{u}_h^{m+1} \in [U_h]^2$ and $p_h^{m+1} \in W_h$ such that

$$
\begin{aligned}
&\int_\Omega \frac{\mathbf{u}_h^{m+1} - \mathbf{u}_h^m}{\triangle t} \cdot \mathbf{v}_h \, dxdy + \int_\Omega (\mathbf{u}_h^{m+1} \cdot \nabla)\mathbf{u}_h^{m+1} \cdot \mathbf{v}_h \, dxdy \\
&+ \int_\Omega 2\nu \mathbb{D}(\mathbf{u}_h^{m+1}) : \mathbb{D}(\mathbf{v}_h) \, dxdy - \int_\Omega p_h^{m+1} (\nabla \cdot \mathbf{v}_h) \, dxdy \\
&= \int_\Omega \mathbf{f}(t_{m+1}) \cdot \mathbf{v}_h \, dxdy, \\
&- \int_\Omega (\nabla \cdot \mathbf{u}_h^{m+1}) q_h \, dxdy = 0,
\end{aligned}
$$

for any $\mathbf{v}_h \in [U_h]^2$ and $q_h \in W_h$.

# Full discretization

- For $m = 0, \cdots, M_m - 1$, find $u_{1h}^{m+1}, \ u_{2h}^{m+1} \in U_h$ and $p_h^{m+1} \in W_h$ such that

$$
\begin{aligned}
&\int_\Omega \frac{u_{1h}^{m+1} - u_{1h}^m}{\triangle t} v_{1h} \ dxdy + \int_\Omega \frac{u_{2h}^{m+1} - u_{2h}^m}{\triangle t} v_{2h} \ dxdy + \int_\Omega \Big( u_{1h}^{m+1} \frac{\partial u_{1h}^{m+1}}{\partial x} v_{1h} \\
&+ u_{2h}^{m+1} \frac{\partial u_{1h}^{m+1}}{\partial y} v_{1h} + u_{1h}^{m+1} \frac{\partial u_{2h}^{m+1}}{\partial x} v_{2h} + u_{2h}^{m+1} \frac{\partial u_{2h}^{m+1}}{\partial y} v_{2h} \Big) \ dxdy \\
&+ \int_\Omega \nu \Big( 2 \frac{\partial u_{1h}^{m+1}}{\partial x} \frac{\partial v_{1h}}{\partial x} + 2 \frac{\partial u_{2h}^{m+1}}{\partial y} \frac{\partial v_{2h}}{\partial y} + \frac{\partial u_{1h}^{m+1}}{\partial y} \frac{\partial v_{1h}}{\partial y} + \frac{\partial u_{1h}^{m+1}}{\partial y} \frac{\partial v_{2h}}{\partial x} \\
&+ \frac{\partial u_{2h}^{m+1}}{\partial x} \frac{\partial v_{1h}}{\partial y} + \frac{\partial u_{2h}^{m+1}}{\partial x} \frac{\partial v_{2h}}{\partial x} \Big) \ dxdy - \int_\Omega \left( p_h^{m+1} \frac{\partial v_{1h}}{\partial x} + p_h^{m+1} \frac{\partial v_{2h}}{\partial y} \right) \ dxdy \\
&= \int_\Omega f_1(t_{m+1}) v_{1h} \ dxdy \int_\Omega f_2(t_{m+1}) v_{2h} \ dxdy, \\
&- \int_\Omega \Big( \frac{\partial u_{1h}^{m+1}}{\partial x} q_h + \frac{\partial u_{2h}^{m+1}}{\partial y} q_h \Big) dxdy = 0,
\end{aligned}
$$

for any $v_{1h} \in U_h$, $v_{2h} \in U_h$, and $q_h \in W_h$.

# Outline

# Newton's iteration

- How to handle the nonlinear terms in the full discretization?
- At each time iteration step of the full discretization, we have a steady nonlinear problem, which is similar to to the steady Navier-Stokes equation.
- Newton's iteration at each time iteration step!
- Given the initial condition $\mathbf{u}_h^0$ and $p_h^0$ at the initial time

# Newton's iteration at each time iteration step

At the $(m+1)^{th}$ step $(m = 0, \cdots, M_m - 1)$ of the time iteration, we consider the following Newton's iteration:

- Initial guess: $\mathbf{u}_h^{m+1,(0)}$ and $p_h^{m+1,(0)}$. Usually they can be the solutions $\mathbf{u}_h^m$ and $p_h^m$ of the previous time iteration step.
- Newton's iteration for full discretization: for $l = 1, 2, \cdots, L$, find $\mathbf{u}_h^{m+1,(l)} \in U_h \times U_h$ and $p_h^{m+1,(l)} \in W_h$ such that

$$\left(\frac{\mathbf{u}_h^{m+1,(l)} - \mathbf{u}_h^m}{\triangle t}, \mathbf{v}\right) + c(\mathbf{u}_h^{m+1,(l)}, \mathbf{u}_h^{m+1,(l-1)}, \mathbf{v}_h)$$
$$+ c(\mathbf{u}_h^{m+1,(l-1)}, \mathbf{u}_h^{m+1,(l)}, \mathbf{v}_h) + a(\mathbf{u}_h^{m+1,(l)}, \mathbf{v}_h) + b(\mathbf{v}_h, p_h^{m+1,(l)})$$
$$= (\mathbf{f}(t_{m+1}), \mathbf{v}_h) + c(\mathbf{u}_h^{m+1,(l-1)}, \mathbf{u}_h^{m+1,(l-1)}, \mathbf{v}_h),$$
$$b(\mathbf{u}_h^{m+1,(l)}, q_h) = 0,$$

  for any $\mathbf{v}_h \in U_h \times U_h$ and $q_h \in W_h$.
- Let $\mathbf{u}_h^{m+1}$ be the final $\mathbf{u}_h^{m+1,(l)}$ from the above iteration.

# Newton's iteration at each time iteration step

- Initial guess: $\mathbf{u}_h^{m+1,(0)}$ and $p_h^{m+1,(0)}$. Usually they can be the solutions $\mathbf{u}_h^m$ and $p_h^m$ of the previous time iteration step.
- Newton's iteration for full discretization in the vector format: for $l = 1, 2, \cdots, L$, find $\mathbf{u}_h^{m+1,(l)} \in U_h \times U_h$ and $p_h^{m+1,(l)} \in W_h$ such that

$$\int_\Omega \frac{\mathbf{u}_h^{m+1,(l)} - \mathbf{u}_h^m}{\triangle t} \cdot \mathbf{v}_h \, dxdy + \int_\Omega (\mathbf{u}_h^{m+1,(l)} \cdot \nabla)\mathbf{u}_h^{m+1,(l-1)} \cdot \mathbf{v}_h \, dxdy$$
$$+ \int_\Omega (\mathbf{u}_h^{m+1,(l-1)} \cdot \nabla)\mathbf{u}_h^{m+1,(l)} \cdot \mathbf{v}_h \, dxdy$$
$$+ \int_\Omega 2\nu \mathbb{D}(\mathbf{u}_h^{m+1,(l)}) : \mathbb{D}(\mathbf{v}_h) \, dxdy - \int_\Omega p_h^{m+1,(l)}(\nabla \cdot \mathbf{v}_h) \, dxdy$$
$$= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \, dxdy + \int_\Omega (\mathbf{u}_h^{m+1,(l-1)} \cdot \nabla)\mathbf{u}_h^{m+1,(l-1)} \cdot \mathbf{v}_h \, dxdy,$$
$$- \int_\Omega (\nabla \cdot \mathbf{u}_h^{m+1,(l)}) q_h \, dxdy = 0,$$

  for any $\mathbf{v}_h \in U_h \times U_h$ and $q_h \in W_h$.
- Let $\mathbf{u}_h^{m+1}$ be the final $\mathbf{u}_h^{m+1,(l)}$ from the above iteration.

# Newton's iteration at each time iteration step

- Initial guess: $u_{1h}^{m+1,(0)}$, $u_{2h}^{m+1,(0)}$, and $p_h^{m+1,(0)}$. Usually they can be the solutions $u_{1h}^m$, $u_{2h}^m$, and $p_h^m$ of the previous time iteration step.

- Newton's iteration for full discretization in the scalar format: for $l = 1, 2, \cdots, L$, find $u_{1h}^{m+1,(l)} \in U_h$, $u_{2h}^{m+1,(l)} \in U_h$, and $p_h^{m+1,(l)} \in W_h$ such that

# Newton's iteration at each time iteration step

$$\int_\Omega \frac{u_{1h}^{m+1,(l)} - u_{1h}^{m}}{\triangle t} v_{1h}\ dxdy + \int_\Omega \frac{u_{2h}^{m+1,(l)} - u_{2h}^{m}}{\triangle t} v_{2h}\ dxdy + \int_\Omega \Big( u_{1h}^{m+1,(l)} \frac{\partial u_{1h}^{m+1,(l-1)}}{\partial x} v_{1h}$$

$$+ u_{2h}^{m+1,(l)} \frac{\partial u_{1h}^{m+1,(l-1)}}{\partial y} v_{1h} + u_{1h}^{m+1,(l)} \frac{\partial u_{2h}^{m+1,(l-1)}}{\partial x} v_{2h} + u_{2h}^{m+1,(l)} \frac{\partial u_{2h}^{m+1,(l-1)}}{\partial y} v_{2h} \Big)\ dxdy$$

$$+ \int_\Omega \Big( u_{1h}^{m+1,(l-1)} \frac{\partial u_{1h}^{m+1,(l)}}{\partial x} v_{1h} + u_{2h}^{m+1,(l-1)} \frac{\partial u_{1h}^{m+1,(l)}}{\partial y} v_{1h} + u_{1h}^{m+1,(l-1)} \frac{\partial u_{2h}^{m+1,(l)}}{\partial x} v_{2h}$$

$$+ u_{2h}^{m+1,(l-1)} \frac{\partial u_{2h}^{m+1,(l)}}{\partial y} v_{2h} \Big)\ dxdy$$

$$+ \int_\Omega \nu \Big( 2 \frac{\partial u_{1h}^{m+1,(l)}}{\partial x} \frac{\partial v_{1h}}{\partial x} + 2 \frac{\partial u_{2h}^{m+1,(l)}}{\partial y} \frac{\partial v_{2h}}{\partial y} + \frac{\partial u_{1h}^{m+1,(l)}}{\partial y} \frac{\partial v_{1h}}{\partial y} + \frac{\partial u_{1h}^{m+1,(l)}}{\partial y} \frac{\partial v_{2h}}{\partial x}$$

$$+ \frac{\partial u_{2h}^{m+1,(l)}}{\partial x} \frac{\partial v_{1h}}{\partial y} + \frac{\partial u_{2h}^{m+1,(l)}}{\partial x} \frac{\partial v_{2h}}{\partial x} \Big)\ dxdy - \int_\Omega \left( p_h^{m+1,(l)} \frac{\partial v_{1h}}{\partial x} + p_h^{m+1,(l)} \frac{\partial v_{2h}}{\partial y} \right)\ dxdy$$

$$= \int_\Omega (f_1 v_{1h} + f_2 v_{2h})\ dxdy + \int_\Omega \Big( u_{1h}^{m+1,(l-1)} \frac{\partial u_{1h}^{m+1,(l-1)}}{\partial x} v_{1h} + u_{2h}^{m+1,(l-1)} \frac{\partial u_{1h}^{m+1,(l-1)}}{\partial y} v_{1h}$$

$$+ u_{1h}^{m+1,(l-1)} \frac{\partial u_{2h}^{m+1,(l-1)}}{\partial x} v_{2h} + u_{2h}^{m+1,(l-1)} \frac{\partial u_{2h}^{m+1,(l-1)}}{\partial y} v_{2h} \Big)\ dxdy,$$

# Newton's iteration at each time iteration step

- Continued formulation:

$$-\int_\Omega \left( \frac{\partial u_{1h}^{m+1,(l)}}{\partial x} q_h + \frac{\partial u_{2h}^{m+1,(l)}}{\partial y} q_h \right) \, dxdy = 0.$$

for any $v_{1h} \in U_h$, $v_{2h} \in U_h$, and $q_h \in W_h$.

- Let $u_{1h}^{m+1}$ and $u_{2h}^{m+1}$ be the final $u_{1h}^{m+1,(l)}$ and $u_{2h}^{m+1,(l)}$ from the above iteration.

# Outline

1. Weak formulation
2. Semi-discretization
3. Full discretization
4. Newton's iteration
5. Matrix formulation
6. FE method
7. More Discussion

# Matrix formulation

- Since $u_{1h}^{m+1,(l)}$, $u_{2h}^{m+1,(l)}$, $u_{1h}^m$, $u_{2h}^m \in U_h = span\{\phi_j\}_{j=1}^{N_b}$ and $p_h^{m+1,(l)}$, $p_h^m \in W_h = span\{\psi_j\}_{j=1}^{N_{bp}}$, then

$$u_{1h}^{m+1,(l)} = \sum_{j=1}^{N_b} u_{1j}^{m+1,(l)}\phi_j, \quad u_{1h}^m = \sum_{j=1}^{N_b} u_{1j}^m\phi_j,$$

$$u_{2h}^{m+1,(l)} = \sum_{j=1}^{N_b} u_{2j}^{m+1,(l)}\phi_j, \quad u_{2h}^m = \sum_{j=1}^{N_b} u_{2j}^m\phi_j$$

$$p_h^{m+1,(l)} = \sum_{j=1}^{N_{bp}} p_j^{m+1,(l)}\psi_j, \quad p_h^m = \sum_{j=1}^{N_{bp}} p_j^m\psi_j,$$

for some coefficients $u_{1j}^{m+1,(l)}$, $u_{2j}^{m+1,(l)}$, $u_{1j}^m$, $u_{2j}^m$ $(j = 1, \cdots, N_b)$, and $p_j^{m+1,(l)}$, $p_j^m$, $(j = 1, \cdots, N_{bp})$.

# Matrix formulation

- If we can set up a linear algebraic system for $u_{1j}^{m+1,(l)}$, $u_{2j}^{m+1,(l)}$ $(j = 1, \cdots, N_b)$, and $p_j^{m+1,(l)}$ $(j = 1, \cdots, N_{bp})$, then we can solve it to obtain the finite element solution $\mathbf{u}_h^{m+1,(l)} = (u_{1h}^{m+1,(l)}, u_{2h}^{m+1,(l)})^t$ and $p_h^{m+1,(l)}$ at the step $l$ $(l = 1, 2, \cdots, L)$ of Newton's iteration.
- For the first equation at the step $l$ $(l = 1, 2, \cdots, L)$ of Newton's iteration, we choose $\mathbf{v}_h = (\phi_i, 0)^t$ $(i = 1, \cdots, N_b)$ and $\mathbf{v}_h = (0, \phi_i)^t$ $(i = 1, \cdots, N_b)$. That is, in the first set of test functions, we choose $v_{1h} = \phi_i$ $(i = 1, \cdots, N_b)$ and $v_{2h} = 0$; in the second set of test functions, we choose $v_{1h} = 0$ and $v_{2h} = \phi_i$ $(i = 1, \cdots, N_b)$.
- For the second equation at the step $l$ $(l = 1, 2, \cdots, L)$ of Newton's iteration, we choose $q_h = \psi_i$ $(i = 1, \cdots, N_{bp})$.

# Matrix formulation

- Set $\mathbf{v}_h = (\phi_i, 0)^t$, i.e., $v_{1h} = \phi_i$ and $v_{2h} = 0$ $(i = 1, \cdots, N_b)$, in the first equation at the step $l$ $(l = 1, 2, \cdots, L)$ of Newton's iteration. Then

$$\frac{1}{\triangle t}\int_\Omega \left(\sum_{j=1}^{N_b} u_{1j}^{m+1,(l)}\phi_j\right)\phi_i\ dxdy - \frac{1}{\triangle t}\int_\Omega \left(\sum_{j=1}^{N_b} u_{1j}^{m}\phi_j\right)\phi_i\ dxdy$$

$$+\int_\Omega \frac{\partial u_{1h}^{m+1,(l-1)}}{\partial x}\left(\sum_{j=1}^{N_b} u_{1j}^{m+1,(l)}\phi_j\right)\phi_i dxdy + \int_\Omega \frac{\partial u_{1h}^{m+1,(l-1)}}{\partial y}\left(\sum_{j=1}^{N_b} u_{2j}^{m+1,(l)}\phi_j\right)\phi_i dxdy$$

$$+\int_\Omega u_{1h}^{m+1,(l-1)}\left(\sum_{j=1}^{N_b} u_{1j}^{m+1,(l)}\frac{\partial \phi_j}{\partial x}\right)\phi_i dxdy + \int_\Omega u_{2h}^{m+1,(l-1)}\left(\sum_{j=1}^{N_b} u_{1j}^{m+1,(l)}\frac{\partial \phi_j}{\partial y}\right)\phi_i dxdy$$

$$+2\int_\Omega \nu\left(\sum_{j=1}^{N_b} u_{1j}^{m+1,(l)}\frac{\partial \phi_j}{\partial x}\right)\frac{\partial \phi_i}{\partial x}dxdy + \int_\Omega \nu\left(\sum_{j=1}^{N_b} u_{1j}^{m+1,(l)}\frac{\partial \phi_j}{\partial y}\right)\frac{\partial \phi_i}{\partial y}dxdy$$

$$+\int_\Omega \nu\left(\sum_{j=1}^{N_b} u_{2j}^{m+1,(l)}\frac{\partial \phi_j}{\partial x}\right)\frac{\partial \phi_i}{\partial y}dxdy - \int_\Omega \left(\sum_{j=1}^{N_{bp}} p_j^{m+1,(l)}\psi_j\right)\frac{\partial \phi_i}{\partial x}dxdy$$

$$= \int_\Omega f_1\phi_i dxdy + \int_\Omega u_{1h}^{m+1,(l-1)}\frac{\partial u_{1h}^{m+1,(l-1)}}{\partial x}\phi_i dxdy + \int_\Omega u_{2h}^{m+1,(l-1)}\frac{\partial u_{1h}^{m+1,(l-1)}}{\partial y}\phi_i dxdy.$$

# Matrix formulation

- Set $\mathbf{v}_h = (0, \phi_i)^t$, i.e., $v_{1h} = 0$ and $v_{2h} = \phi_i$ $(i = 1, \cdots, N_b)$, in the first equation of the Galerkin formulation at the step $l$ $(l = 1, 2, \cdots, L)$ of Newton's iteration. Then

$$
\begin{aligned}
&\frac{1}{\triangle t}\int_\Omega (\sum_{j=1}^{N_b} u_{2j}^{m+1,(l)}\phi_j)\phi_i \; dxdy - \frac{1}{\triangle t}\int_\Omega (\sum_{j=1}^{N_b} u_{2j}^{m}\phi_j)\phi_i \; dxdy \\
&+ \int_\Omega \frac{\partial u_{2h}^{m+1,(l-1)}}{\partial x}(\sum_{j=1}^{N_b} u_{1j}^{m+1,(l)}\phi_j)\phi_i dxdy + \int_\Omega \frac{\partial u_{2h}^{m+1,(l-1)}}{\partial y}(\sum_{j=1}^{N_b} u_{2j}^{m+1,(l)}\phi_j)\phi_i dxdy \\
&+ \int_\Omega u_{1h}^{m+1,(l-1)}(\sum_{j=1}^{N_b} u_{2j}^{m+1,(l)}\frac{\partial \phi_j}{\partial x})\phi_i dxdy + \int_\Omega u_{2h}^{m+1,(l-1)}(\sum_{j=1}^{N_b} u_{2j}^{m+1,(l)}\frac{\partial \phi_j}{\partial y})\phi_i dxdy \\
&+2\int_\Omega \nu(\sum_{j=1}^{N_b} u_{2j}^{m+1,(l)}\frac{\partial \phi_j}{\partial y})\frac{\partial \phi_i}{\partial y} dxdy + \int_\Omega \nu(\sum_{j=1}^{N_b} u_{1j}^{m+1,(l)}\frac{\partial \phi_j}{\partial y})\frac{\partial \phi_i}{\partial x} dxdy \\
&+ \int_\Omega \nu(\sum_{j=1}^{N_b} u_{2j}^{m+1,(l)}\frac{\partial \phi_j}{\partial x})\frac{\partial \phi_i}{\partial x} dxdy - \int_\Omega (\sum_{j=1}^{N_{bp}} p_j^{m+1,(l)}\psi_j)\frac{\partial \phi_i}{\partial y} dxdy \\
= &\int_\Omega f_2\phi_i dxdy + \int_\Omega u_{1h}^{m+1,(l-1)}\frac{\partial u_{2h}^{m+1,(l-1)}}{\partial x}\phi_i dxdy + \int_\Omega u_{2h}^{m+1,(l-1)}\frac{\partial u_{2h}^{m+1,(l-1)}}{\partial y}\phi_i dxdy.
\end{aligned}
$$

# Matrix formulation

- Set $q_h = \psi_i$ $(i = 1, \cdots, N_{bp})$ in the second equation of the Galerkin formulation at the step $l$ $(l = 1, 2, \cdots, L)$ of Newton's iteration. Then

$$-\int_\Omega \left( \sum_{j=1}^{N_b} u_{1j}^{m+1,(l)} \frac{\partial \phi_j}{\partial x} \right) \psi_i \ dxdy - \int_\Omega \left( \sum_{j=1}^{N_b} u_{2j}^{m+1,(l)} \frac{\partial \phi_j}{\partial y} \right) \psi_i \ dxdy = 0.$$

# Matrix formulation

- Simplify the above three sets of equations, we obtain

$$
\begin{aligned}
&\sum_{j=1}^{N_b} u_{1j}^{m+1,(l)}\Big(\frac{1}{\triangle t}\int_\Omega \phi_j\phi_i\ dxdy + 2\int_\Omega \nu\frac{\partial\phi_j}{\partial x}\frac{\partial\phi_i}{\partial x}\ dxdy + \int_\Omega \nu\frac{\partial\phi_j}{\partial y}\frac{\partial\phi_i}{\partial y}\ dxdy \\
&+\int_\Omega \frac{\partial u_{1h}^{m+1,(l-1)}}{\partial x}\phi_j\phi_i\ dxdy + \int_\Omega u_{1h}^{m+1,(l-1)}\frac{\partial\phi_j}{\partial x}\phi_i\ dxdy \\
&+\int_\Omega u_{2h}^{m+1,(l-1)}\frac{\partial\phi_j}{\partial y}\phi_i\ dxdy\Big) + \sum_{j=1}^{N_b} u_{2j}^{m+1,(l)}\Big(\int_\Omega \nu\frac{\partial\phi_j}{\partial x}\frac{\partial\phi_i}{\partial y}\ dxdy \\
&+\int_\Omega \frac{\partial u_{1h}^{m+1,(l-1)}}{\partial y}\phi_j\phi_i\ dxdy\Big) + \sum_{j=1}^{N_{bp}} p_j^{m+1,(l)}\left(-\int_\Omega \psi_j\frac{\partial\phi_i}{\partial x}\ dxdy\right) \\
&=\int_\Omega f_1\phi_i dxdy + \int_\Omega u_{1h}^{m+1,(l-1)}\frac{\partial u_{1h}^{m+1,(l-1)}}{\partial x}\phi_i\ dxdy \\
&+\int_\Omega u_{2h}^{m+1,(l-1)}\frac{\partial u_{1h}^{m+1,(l-1)}}{\partial y}\phi_i\ dxdy + \sum_{j=1}^{N_b} u_{1j}^{m}\left(\frac{1}{\triangle t}\int_\Omega \phi_j\phi_i\ dxdy\right),
\end{aligned}
$$

# Matrix formulation

- Continued formulation:

$$\sum_{j=1}^{N_b} u_{1j}^{m+1,(l)} \left( \int_\Omega \nu \frac{\partial \phi_j}{\partial y} \frac{\partial \phi_i}{\partial x} \, dxdy + \int_\Omega \frac{\partial u_{2h}^{m+1,(l-1)}}{\partial x} \phi_j \phi_i \, dxdy \right)$$

$$+ \sum_{j=1}^{N_b} u_{2j}^{m+1,(l)} \Big( \frac{1}{\triangle t} \int_\Omega \phi_j \phi_i \, dxdy + 2 \int_\Omega \nu \frac{\partial \phi_j}{\partial y} \frac{\partial \phi_i}{\partial y} \, dxdy$$

$$+ \int_\Omega \nu \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial x} \, dxdy + \int_\Omega \frac{\partial u_{2h}^{m+1,(l-1)}}{\partial y} \phi_j \phi_i \, dxdy + \int_\Omega u_{1h}^{m+1,(l-1)} \frac{\partial \phi_j}{\partial x} \phi_i \, dxdy$$

$$+ \int_\Omega u_{2h}^{m+1,(l-1)} \frac{\partial \phi_j}{\partial y} \phi_i \, dxdy \Big) + \sum_{j=1}^{N_{bp}} p_j^{m+1,(l)} \left( - \int_\Omega \psi_j \frac{\partial \phi_i}{\partial y} \, dxdy \right)$$

$$= \int_\Omega f_2 \phi_i dxdy + \int_\Omega u_{1h}^{m+1,(l-1)} \frac{\partial u_{2h}^{m+1,(l-1)}}{\partial x} \phi_i \, dxdy$$

$$+ \int_\Omega u_{2h}^{m+1,(l-1)} \frac{\partial u_{2h}^{m+1,(l-1)}}{\partial y} \phi_i \, dxdy + \sum_{j=1}^{N_b} u_{2j}^m \left( \frac{1}{\triangle t} \int_\Omega \phi_j \phi_i \, dxdy \right),$$

# Matrix formulation

- Continued formulation:

$$
\begin{aligned}
& \sum_{j=1}^{N_b} u_{1j}^{m+1,(l)} \left( -\int_\Omega \frac{\partial \phi_j}{\partial x} \psi_i \ dxdy \right) \\
& + \sum_{j=1}^{N_b} u_{2j}^{m+1,(l)} \left( -\int_\Omega \frac{\partial \phi_j}{\partial y} \psi_i \ dxdy \right) + \sum_{j=1}^{N_{bp}} p_j^{m+1,(l)} * 0 \\
= \ & 0.
\end{aligned}
$$

# Matrix formulation

- Define

$$A_1 = \left[\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial x}\, dxdy\right]_{i,j=1}^{N_b}, \quad A_2 = \left[\int_\Omega \nu \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial y}\, dxdy\right]_{i,j=1}^{N_b},$$

$$A_3 = \left[\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial y}\, dxdy\right]_{i,j=1}^{N_b}, \quad A_4 = \left[\int_\Omega \nu \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial x}\, dxdy\right]_{i,j=1}^{N_b},$$

$$A_5 = \left[\int_\Omega -\psi_j \frac{\partial \phi_i}{\partial x}\, dxdy\right]_{i=1,j=1}^{N_b,N_{bp}}, \quad A_6 = \left[\int_\Omega -\psi_j \frac{\partial \phi_i}{\partial y}\, dxdy\right]_{i=1,j=1}^{N_b,N_{bp}},$$

$$A_7 = \left[\int_\Omega -\frac{\partial \phi_j}{\partial x}\psi_i\, dxdy\right]_{i=1,j=1}^{N_{bp},N_b}, \quad A_8 = \left[\int_\Omega -\frac{\partial \phi_j}{\partial y}\psi_i\, dxdy\right]_{i=1,j=1}^{N_{bp},N_b}.$$

- Define a zero matrix $\mathbb{O}_1 = [0]_{i=1,j=1}^{N_{bp},N_{bp}}$ whose size is $N_{bp} \times N_{bp}$. Then

$$A = \begin{pmatrix} 2A_1 + A_2 & A_3 & A_5 \\ A_4 & 2A_2 + A_1 & A_6 \\ A_7 & A_8 & \mathbb{O}_1 \end{pmatrix}$$

# Matrix formulation

- Each matrix above can be obtained by Algorithm I-3 in Chapter 3.
- It is not difficult to verify (an independent study project topic) that

$$A_4 = A_3^t, \quad A_7 = A_5^t, \quad A_8 = A_6^t.$$

- Hence the matrix $A$ is actually symmetric:

$$A = \begin{pmatrix} 2A_1 + A_2 & A_3 & A_5 \\ A_3^t & 2A_2 + A_1 & A_6 \\ A_5^t & A_6^t & \mathbb{O}_1 \end{pmatrix}$$

# Another format of full discretization

- Define the basic mass matrix

$$M_e = [m_{ij}]_{i,j=1}^{N_b} = \left[ \int_\Omega \phi_j \phi_i \, dxdy \right]_{i,j=1}^{N_b}.$$

- The mass matrix $M_e$ can be obtained by Algorithm I-3 in Chapter 3, with $r = s = p = q = 0$ and $c = 1$.
- Define zero matrices $\mathbb{O}_2 = [0]_{i=1,j=1}^{N_b,N_{bp}}$ and $\mathbb{O}_3 = [0]_{i=1,j=1}^{N_b,N_b}$. Then define the block mass matrix

$$M = \begin{pmatrix} M_e & \mathbb{O}_3 & \mathbb{O}_2 \\ \mathbb{O}_3 & M_e & \mathbb{O}_2 \\ \mathbb{O}_2^t & \mathbb{O}_2^t & \mathbb{O}_1 \end{pmatrix}$$

# Matrix formulation

- Define

$$AN_1 = \left[\int_\Omega \frac{\partial u_{1h}^{m+1,(l-1)}}{\partial x}\phi_j\phi_i \ dxdy\right]_{i,j=1}^{N_b}, \quad AN_2 = \left[\int_\Omega u_{1h}^{m+1,(l-1)}\frac{\partial \phi_j}{\partial x}\phi_i \ dxdy\right]_{i,j=1}^{N_b}$$

$$AN_3 = \left[\int_\Omega u_{2h}^{m+1,(l-1)}\frac{\partial \phi_j}{\partial y}\phi_i \ dxdy\right]_{i,j=1}^{N_b}, \quad AN_4 = \left[\int_\Omega \frac{\partial u_{1h}^{m+1,(l-1)}}{\partial y}\phi_j\phi_i \ dxdy\right]_{i,j=1}^{N_b}$$

$$AN_5 = \left[\int_\Omega \frac{\partial u_{2h}^{m+1,(l-1)}}{\partial x}\phi_j\phi_i \ dxdy\right]_{i,j=1}^{N_b}, \quad AN_6 = \left[\int_\Omega \frac{\partial u_{2h}^{m+1,(l-1)}}{\partial y}\phi_j\phi_i \ dxdy\right]_{i,j=1}^{N_b}$$

- Then

$$AN = \begin{pmatrix} AN_1 + AN_2 + AN_3 & AN_4 & \mathbb{O}_2 \\ AN_5 & AN_6 + AN_2 + AN_3 & \mathbb{O}_2 \\ \mathbb{O}_2^t & \mathbb{O}_2^t & \mathbb{O}_1 \end{pmatrix}$$

- Each matrix above can be obtained by Algorithm VIII in Chapter 7.

# Matrix formulation

- Define the load vector

$$\vec{b} = \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{0} \end{pmatrix}$$

where

$$\vec{b}_1 = \left[\int_\Omega f_1 \phi_i dxdy\right]_{i=1}^{N_b}, \quad \vec{b}_2 = \left[\int_\Omega f_2 \phi_i dxdy\right]_{i=1}^{N_b}.$$

Here the size of the zero vector is $N_{bp} \times 1$. That is, $\vec{0} = [0]_{i=1}^{N_{bp}}$.

- Each of $\vec{b}_1$ and $\vec{b}_2$ can be obtained by Algorithm II-5 in Chapter 4.

# Matrix formulation

- Define the vector

$$\overrightarrow{bN} = \begin{pmatrix} \overrightarrow{bN}_1 + \overrightarrow{bN}_2 \\ \overrightarrow{bN}_3 + \overrightarrow{bN}_4 \\ \vec{0} \end{pmatrix}$$

where $\vec{0} = [0]_{i=1}^{N_{bp}}$ and

$$\overrightarrow{bN}_1 = \left[\int_\Omega u_{1h}^{m+1,(l-1)} \frac{\partial u_{1h}^{m+1,(l-1)}}{\partial x} \phi_i \ dxdy\right]_{i=1}^{N_b},$$

$$\overrightarrow{bN}_2 = \left[\int_\Omega u_{2h}^{m+1,(l-1)} \frac{\partial u_{1h}^{m+1,(l-1)}}{\partial y} \phi_i \ dxdy\right]_{i=1}^{N_b},$$

$$\overrightarrow{bN}_3 = \left[\int_\Omega u_{1h}^{m+1,(l-1)} \frac{\partial u_{2h}^{m+1,(l-1)}}{\partial x} \phi_i \ dxdy\right]_{i=1}^{N_b},$$

$$\overrightarrow{bN}_4 = \left[\int_\Omega u_{2h}^{m+1,(l-1)} \frac{\partial u_{2h}^{m+1,(l-1)}}{\partial y} \phi_i \ dxdy\right]_{i=1}^{N_b}.$$

# Matrix formulation

- Each vector above can be obtained by Algorithm IX in Chapter 7.
- Define the known vector from the previous time iteration step:

$$\overrightarrow{X}^m = \begin{pmatrix} \overrightarrow{X}_1^m \\ \overrightarrow{X}_2^m \\ \overrightarrow{X}_3^m \end{pmatrix}$$

where

$$\vec{X}_1^m = \left[u_{1j}^m\right]_{j=1}^{N_b},$$
$$\vec{X}_2^m = \left[u_{2j}^m\right]_{j=1}^{N_b},$$
$$\vec{X}_3^m = \left[p_j^m\right]_{j=1}^{N_{bp}}.$$

# Matrix formulation

- Define the unknown vector

$$\vec{X}^{m+1,(l)} = \begin{pmatrix} \vec{X}_1^{m+1,(l)} \\ \vec{X}_2^{m+1,(l)} \\ \vec{X}_3^{m+1,(l)} \end{pmatrix}$$

where

$$\vec{X}_1^{m+1,(l)} = \left[u_{1j}^{m+1,(l)}\right]_{j=1}^{N_b},$$
$$\vec{X}_2^{m+1,(l)} = \left[u_{2j}^{m+1,(l)}\right]_{j=1}^{N_b},$$
$$\vec{X}_3^{m+1,(l)} = \left[p_{j}^{m+1,(l)}\right]_{j=1}^{N_{bp}}.$$

# Matrix formulation

- Define

$$A^{m+1,(l)} = \frac{M}{\triangle t} + A + AN, \quad \vec{b}^{m+1,(l)} = \vec{b} + \frac{M}{\triangle t}\overrightarrow{X}^{m} + \overrightarrow{bN}.$$

- For step $l$ ($l = 1, 2, \cdots, L$) of the Newton's iteration at the $(m+1)^{th}$ step of the time iteration, we obtain the linear algebraic system

$$A^{m+1,(l)}\vec{X}^{m+1,(l)} = \vec{b}^{m+1,(l)}.$$

- Let $X^{m+1}$ be the final $\vec{X}^{m+1,(l)}$ from the above Newton's iteration at the $(m+1)^{th}$ step of the time iteration.

# Outline

# Assembly of a time-independent matrix

Recall Algorithm I-3 from Chapter 3:

- Initialize the matrix: $A = sparse(N_b^{test}, N_b^{trial})$;
- Compute the integrals and assemble them into $A$:

$FOR\ n = 1, \cdots, N$

$\quad FOR\ \alpha = 1, \cdots, N_{lb}^{trial}$

$\quad\quad FOR\ \beta = 1, \cdots, N_{lb}^{test}$

$\quad\quad\quad$ Compute $r = \int_{E_n} c \frac{\partial^{r+s}\varphi_{n\alpha}}{\partial x^r \partial y^s} \frac{\partial^{p+q}\psi_{n\beta}}{\partial x^p \partial y^q}\ dxdy$;

$\quad\quad\quad$ Add $r$ to $A(T_b^{test}(\beta, n), T_b^{trial}(\alpha, n))$.

$\quad\quad END$

$\quad END$

$END$

# Assembly of the time-independent stiffness matrix

- Call Algorithm I-3 with $r = 1, \ s = 0, \ p = 1, \ q = 0, \ c = \nu$ , basis type of $\mathbf{u}$ for trial function, and basis type of $\mathbf{u}$ for test function, to obtain $A_1$.
- Call Algorithm I-3 with $r = 0, \ s = 1, \ p = 0, \ q = 1, \ c = \nu$ , basis type of $\mathbf{u}$ for trial function, and basis type of $\mathbf{u}$ for test function, to obtain $A_2$.
- Call Algorithm I-3 with $r = 1, \ s = 0, \ p = 0, \ q = 1, \ c = \nu$ , basis type of $\mathbf{u}$ for trial function, and basis type of $\mathbf{u}$ for test function, to obtain $A_3$.
- Call Algorithm I-3 with $r = 0, \ s = 0, \ p = 1, \ q = 0, \ c = -1$ , basis type of $p$ for trial function, and basis type of $\mathbf{u}$ for test function, to obtain $A_5$.
- Call Algorithm I-3 with $r = 0, \ s = 0, \ p = 0, \ q = 1, \ c = -1$ , basis type of $p$ for trial function, and basis type of $\mathbf{u}$ for test function, to obtain $A_6$.
- Generate a zero matrix $\mathbb{O}$ whose size is $N_{bp} \times N_{bp}$.
- Then the stiffness matrix
  $A = [2A_1 + A_2 \quad A_3 \quad A_5; A_3^t \quad 2A_2 + A_1 \quad A6; A_5^t \quad A_6^t \quad \mathbb{O}]$.

# Assembly of the mass matrix

- Call Algorithm I-3 with $r = 0,\ s = 0,\ p = 0,\ q = 0,\ c = 1$ , basis type of **u** for trial function, and basis type of **u** for test function, to obtain the basic mass matrix $M_e$.
- Generate three zero matrices $\mathbb{O}_1$, $\mathbb{O}_2$, and $\mathbb{O}_3$ whose sizes are $N_{bp} \times N_{bp}$, $N_b \times N_{bp}$, and $N_b \times N_b$, respectively.
- Then the block mass matrix
  $M = [M_e \ \ \mathbb{O}_3 \ \ \mathbb{O}_2; \mathbb{O}_3 \ \ M_e \ \ \mathbb{O}_2; \mathbb{O}_2^t \ \ \mathbb{O}_2^t \ \ \mathbb{O}_1]$.

# Assembly of a time-independent vector

Recall Algorithm II-3 from Chapter 3:

- Initialize the matrix: $b = sparse(N_b, 1)$;
- Compute the integrals and assemble them into $b$:

$FOR\ n = 1, \cdots, N$:

$\quad FOR\ \beta = 1, \cdots, N_{lb}$:

$\quad\quad$ Compute $r = \int_{E_n} f \frac{\partial^{p+q}\psi_{n\beta}}{\partial x^p \partial y^q}\ dxdy$;

$\quad\quad b(T_b(\beta, n), 1) = b(T_b(\beta, n), 1) + r$;

$\quad END$

$END$

# Assembly of a time-dependent vector

Recall Algorithm II-5 from Chapter 4:

- Specify a value for the time $t$ based on the input time;
- Initialize the vector: $b = sparse(N_b, 1)$;
- Compute the integrals and assemble them into $b$:

$FOR\ n = 1, \cdots, N$:

$\quad FOR\ \beta = 1, \cdots, N_{lb}$:

$\quad\quad$ Compute $r = \int_{E_n} f(t) \frac{\partial^{p+q}\psi_{n\beta}}{\partial x^p \partial y^q}\, dxdy$;

$\quad\quad b(T_b(\beta, n), 1) = b(T_b(\beta, n), 1) + r$;

$\quad END$

$END$

# Assembly of the load vector

- Call Algorithm II-5 with $p = q = 0$ and $f = f_1$ to obtain $b_1(t)$.
- Call Algorithm II-5 with $p = q = 0$ and $f = f_2$ to obtain $b_2(t)$.
- Define a zero column vector $\vec{0}$ whose size is $N_{bp} \times 1$.
- Then the load vector $\vec{b} = [b_1(t); b_2(t); \vec{0}]$.
- If $f_1$ and $f_2$ do not depend on $t$, then this part is exactly the same as the assembly of the load vector with Algorithm II-3 in Chapter 7.

# Assembly of a matrix for an integral with a finite element coefficient function

Recall Algorithm VIII from Chapter 7:

- Initialize the matrix: $A = sparse(N_b^{test}, N_b^{trial})$;
- Compute the integrals and assemble them into $A$:

$FOR\ n = 1, \cdots, N$:

$\quad FOR\ \alpha = 1, \cdots, N_{lb}^{trial}$:

$\quad\quad FOR\ \beta = 1, \cdots, N_{lb}^{test}$:

$\quad\quad\quad$ Compute $r = \int_{E_n} \frac{\partial^{d+e} c_h}{\partial x^d \partial y^e} \frac{\partial^{r+s} \varphi_{n\alpha}}{\partial x^r \partial y^s} \frac{\partial^{p+q} \psi_{n\beta}}{\partial x^p \partial y^q}\ dxdy$;

$\quad\quad\quad$ Add $r$ to $A(T_b(\beta, n), T_b(\alpha, n))$.

$\quad\quad END$

$\quad END$

$END$

# Assembly of a matrix for an integral with a finite element coefficient function

- Call Algorithm VIII with $d = 1$, $e = 0$, $r = 0$, $s = 0$, $p = 0$, $q = 0$, $c_h = u_{1h}^{(l-1)}$, basis type of $\mathbf{u}$ for both trial and test functions, to obtain $AN_1$.
- Call Algorithm VIII with $d = 0$, $e = 0$, $r = 1$, $s = 0$, $p = 0$, $q = 0$, $c_h = u_{1h}^{(l-1)}$, basis type of $\mathbf{u}$ for both trial and test functions, to obtain $AN_2$.
- Call Algorithm VIII with $d = 0$, $e = 0$, $r = 0$, $s = 1$, $p = 0$, $q = 0$, $c_h = u_{2h}^{(l-1)}$, basis type of $\mathbf{u}$ for both trial and test functions, to obtain $AN_3$.
- Call Algorithm VIII with $d = 0$, $e = 1$, $r = 0$, $s = 0$, $p = 0$, $q = 0$, $c_h = u_{1h}^{(l-1)}$, basis type of $\mathbf{u}$ for both trial and test functions, to obtain $AN_4$.

# Assembly of a matrix for an integral with a finite element coefficient function

- Call Algorithm VIII with $d = 1$, $e = 0$, $r = 0$, $s = 0$, $p = 0$, $q = 0$, $c_h = u_{2h}^{(l-1)}$, basis type of $\mathbf{u}$ for both trial and test functions, to obtain $AN_5$.
- Call Algorithm VIII with $d = 0$, $e = 1$, $r = 0$, $s = 0$, $p = 0$, $q = 0$, $c_h = u_{2h}^{(l-1)}$, basis type of $\mathbf{u}$ for both trial and test functions, to obtain $AN_6$.
- Generate a zero matrix $\mathbb{O}_1 = [0]_{i,j=1}^{N_{bp}}$, $\mathbb{O}_2 = [0]_{i=1,j=1}^{N_b,N_{bp}}$ and $\mathbb{O}_3 = [0]_{i=1,j=1}^{N_b,N_{bp}}$.
- Then the stiffness matrix

$$A = [AN_1 + AN_2 + AN_3 \;\; AN_4 \;\; \mathbb{O}_2; AN_5 \;\; AN_6 + AN_2 + AN_3 \;\; \mathbb{O}_3; \mathbb{O}_2^t \;\; \mathbb{O}_3^t \;\; \mathbb{O}_1].$$

# Assembly of the vector for an integral with two finite element coefficient functions

Recall Algorithm IX from Chapter 7:

- Initialize the vector: $b = sparse(N_b, 1)$;
- Compute the integrals and assemble them into $b$:

$FOR\ n = 1, \cdots, N$:

$\quad FOR\ \beta = 1, \cdots, N_{lb}$:

$\quad\quad$ Compute $r = \int_{E_n} \frac{\partial^{d+e} f_{1h}}{\partial x^d \partial y^e} \frac{\partial^{r+s} f_{2h}}{\partial x^r \partial y^s} \frac{\partial^{p+q} \psi_{n\beta}}{\partial x^p \partial y^q}\ dxdy$;

$\quad\quad b(T_b(\beta, n), 1) = b(T_b(\beta, n), 1) + r$;

$\quad END$

$END$

# Assembly of the vector for an integral with two finite element coefficient functions

- Call Algorithm IX with $d = 0$, $e = 0$, $r = 1$, $s = 0$, $p = 0$, $q = 0$ and $f_{h1} = u_{1h}^{(l-1)}$, $f_{h2} = u_{1h}^{(l-1)}$ to obtain $bN_1$.
- Call Algorithm IX with $d = 0$, $e = 0$, $r = 0$, $s = 1$, $p = 0$, $q = 0$ and $f_{h1} = u_{2h}^{(l-1)}$, $f_{h2} = u_{1h}^{(l-1)}$ to obtain $bN_2$.
- Call Algorithm IX with $d = 0$, $e = 0$, $r = 1$, $s = 0$, $p = 0$, $q = 0$ and $f_{h1} = u_{1h}^{(l-1)}$, $f_{h2} = u_{2h}^{(l-1)}$ to obtain $bN_3$.
- Call Algorithm IX with $d = 0$, $e = 0$, $r = 0$, $s = 1$, $p = 0$, $q = 0$ and $f_{h1} = u_{2h}^{(l-1)}$, $f_{h2} = u_{2h}^{(l-1)}$ to obtain $bN_4$.
- Define a zero column vector $\vec{0}$ whose size is $N_{bp} \times 1$
- Then the load vector $\overrightarrow{bN} = [bN_1 + bN2; bN_3 + bN_4; \vec{0}]$.

# Time-dependent Dirichlet boundary condition

Recall Algorithm III-4 from Chapter 8:

- Specify a value for the time $t$ based on the input time;
- Deal with the Dirichlet boundary conditions:

*FOR* $k = 1, \cdots, nbn$:

If $boundarynodes(1, k)$ shows Dirichlet condition, then

$i = boundarynodes(2, k)$;

$\bar{A}(i, :) = 0$;

$\bar{A}(i, i) = 1$;

$\bar{b}(i) = g_1(P_b(:, i), t)$;

$\bar{A}(N_b + i, :) = 0$;

$\bar{A}(N_b + i, N_b + i) = 1$;

$\bar{b}(N_b + i) = g_2(P_b(:, i), t)$;

*ENDIF*

*END*

# Main pseudo code

Algorithm $B$:

- Generate the mesh information matrices $P$ and $T$.
- Assemble the mass matrix $M$ and stiffness matrix $A$ by using Algorithm I-3.
- Generate the initial vector $\vec{X}^0$.
- Iterate in time: $FOR\ m = 0, \cdots, M_m - 1$
- $t_{m+1} = (m+1)\triangle t$;
- Assemble the load vector $\vec{b}$ by using Algorithm II-5.
- Newton iteration: $FOR\ l = 1, 2, \cdots, L$
- Assemble the matrix $AN$ by using Algorithm VIII.
- Assemble the vector $\overrightarrow{bN}$ by using Algorithm IX.
- $A^{m+1,(l)} = \frac{M}{\triangle t} + A + AN$ and $\vec{b}^{m+1,(l)} = \vec{b} + \frac{M}{\triangle t}\vec{X}^m + \overrightarrow{bN}$
- Treat Dirichlet boundary for $A^{m+1,(l)}$ and $\vec{b}^{m+1,(l)}$ by Algorithm III-4.
- Solve $A^{m+1,(l)}\vec{X}^{m+1,(l)} = \vec{b}^{m+1,(l)}$ for $\vec{X}$.

  $END$
- Let $X^{m+1}$ be the final $\vec{X}^{m+1,(l)}$ from the above Newton's iteration.

$END$

# Numerical example

- Example 1: On the domain $\Omega = [0,1] \times [-0.25, 0]$, consider the time-dependent Navier-Stokes equation

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla \cdot \mathbb{T}(\mathbf{u}, p) = \mathbf{f} \quad \text{in } \Omega \times [0,1],$$
$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \times [0,1].$$

# Numerical example

Independent study topic:

- (1) Following the traditional way, which was used to set up the numerical examples in the previous chapters, determine the source term $\mathbf{f}$, initial condition, Dirichlet boundary conditions, and fixed value of $p$ at $(0,0)$ such that the analytic solutions of this problem are

$$
\begin{aligned}
u_1 &= (x^2y^2 + e^{-y})cos(2\pi t),\\
u_2 &= \left[-\frac{2}{3}xy^3 + 2 - \pi\sin(\pi x)\right] cos(2\pi t),\\
p &= -[2 - \pi\sin(\pi x)]\cos(2\pi y)cos(2\pi t).
\end{aligned}
$$

- (2)Choose $h = 1/8,\ 1/16,\ 1/32$ and $\triangle t = 8h^3$. Use the Taylor-Hood finite elements with backward Euler scheme to solve this equation and provide the numerical errors of $\mathbf{u}$ and $p$ in $L^2$, $L^\infty$, and $H^1$ norms.

# Outline

# Mixed boundary conditions

- The treatment of the stress/Robin boundary conditions is similar to that of Chapter 8.
- If the functions in the stress/Robin boundary conditions are independent of time, then the same subroutines from Chapter 7 can be used before the time iteration starts.
- If the functions in the stress/Robin boundary conditions depend on time, then the same algorithms as those in Chapter 7 can be used at each time iteration step. But the time needs to be specified in these algorithms.

# Mixed boundary conditions

- Consider

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla \cdot \mathbb{T}(\mathbf{u}, p) = \mathbf{f} \quad \text{in } \Omega \times [0, T],$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \times [0, T],$$

$$\mathbb{T}(\mathbf{u}, p)\mathbf{n} = \mathbf{p} \quad \text{on } \Gamma_S \times [0, T],$$

$$\mathbb{T}(\mathbf{u}, p)\mathbf{n} + r\mathbf{u} = \mathbf{q} \quad \text{on } \Gamma_R \times [0, T],$$

$$\mathbf{u} = \mathbf{g} \quad \text{on } \Gamma_D \times [0, T],$$

$$\mathbf{u} = \mathbf{u}_0, \quad \text{at } t = 0 \text{ and in } \Omega.$$

where $\Gamma_S$, $\Gamma_R \subset \partial\Omega$ and $\Gamma_D = \partial\Omega / (\Gamma_S \cup \Gamma_R)$.

# Mixed boundary conditions

- Recall

$$\int_\Omega \mathbf{u}_t \cdot \mathbf{v}\ dxdy + \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v}\ dxdy + \int_\Omega 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v})\ dxdy$$
$$- \int_\Omega p(\nabla \cdot \mathbf{v})\ dxdy - \int_{\partial\Omega} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v}\ ds \ = \ \int_\Omega \mathbf{f} \cdot \mathbf{v}\ dxdy,$$
$$- \int_\Omega (\nabla \cdot \mathbf{u})q\ dxdy = 0.$$

- Since the solution on $\Gamma_D = \partial\Omega/(\Gamma_S \cup \Gamma_R)$ is given by $\mathbf{u} = \mathbf{g}$, then we can choose the test function $\mathbf{v}(x, y)$ such that $\mathbf{v} = 0$ on $\partial\Omega/(\Gamma_S \cup \Gamma_R)$.

# Mixed boundary conditions

- Hence, similar to the treatment of the mixed boundary condition in Chapter 7, the weak formulation is to find $\mathbf{u} \in H^1(0,T;[H^1(\Omega)]^2)$ and $p \in L^2(0,T;L^2(\Omega))$ such that

$$\int_\Omega \mathbf{u}_t \cdot \mathbf{v}\ dxdy + \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\ dxdy$$

$$+\int_\Omega 2\nu\mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v})\ dxdy - \int_\Omega p(\nabla\cdot\mathbf{v})\ dxdy + \int_{\Gamma_R} r\mathbf{u}\cdot\mathbf{v}\ ds$$

$$= \int_\Omega \mathbf{f}\cdot\mathbf{v}\ dxdy + \int_{\Gamma_R} \mathbf{q}\cdot\mathbf{v}\ ds + \int_{\Gamma_S} \mathbf{p}\cdot\mathbf{v}\ ds,$$

$$-\int_\Omega (\nabla\cdot\mathbf{u})q\ dxdy = 0.$$

  for any $\mathbf{v} \in [H^1_{0D}(\Omega)]^2$ and $q \in L^2(\Omega)$ where $H^1_{0D}(\Omega) = \{w \in H^1(\Omega) : w = 0 \text{ on } \Gamma_D\}$.
- Code? Combine all of the subroutines for Dirichlet/Stress/Robin boundary conditions.

# Mixed boundary conditions in normal/tangential directions

- Consider

$$
\begin{aligned}
&\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla \cdot \mathbb{T}(\mathbf{u}, p) = \mathbf{f} \quad \text{in } \Omega \times [0, T], \\
&\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \times [0, T], \\
&\mathbf{n}^t \mathbb{T}(\mathbf{u}, p)\mathbf{n} = p_n, \tau^t \mathbb{T}(\mathbf{u}, p)\mathbf{n} = p_\tau \text{ on } \Gamma_S \times [0, T], \\
&\mathbf{n}^t \mathbb{T}(\mathbf{u}, p)\mathbf{n} + r\mathbf{n}^t\mathbf{u} = q_n, \tau^t \mathbb{T}(\mathbf{u}, p)\mathbf{n} + r\tau^t\mathbf{u} = q_\tau \text{ on } \Gamma_R \times [0, T], \\
&\mathbf{u} = \mathbf{g} \quad \text{on } \Gamma_D \times [0, T], \\
&\mathbf{u} = \mathbf{u}_0, \quad \text{at } t = 0 \text{ and in } \Omega.
\end{aligned}
$$

where $\Gamma_S,\ \Gamma_R \subset \partial\Omega$, $\Gamma_D = \partial\Omega / (\Gamma_S \cup \Gamma_R)$, $\mathbf{n} = (n_1,\ n_2)^t$ is the unit outer normal vector of $\partial\Omega$, and $\tau = (\tau_1, \tau_2)^t$ is the corresponding unit tangential vector of $\partial\Omega$.

# Mixed boundary conditions in normal/tangential directions

- Recall

$$
\begin{aligned}
&\int_\Omega \mathbf{u}_t \cdot \mathbf{v}\ dxdy + \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\ dxdy + \int_\Omega 2\nu\mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v})\ dxdy \\
&- \int_\Omega p(\nabla\cdot\mathbf{v})\ dxdy - \int_{\partial\Omega} (\mathbb{T}(\mathbf{u},p)\mathbf{n})\cdot\mathbf{v}\ ds \ = \ \int_\Omega \mathbf{f}\cdot\mathbf{v}\ dxdy, \\
&- \int_\Omega (\nabla\cdot\mathbf{u})q\ dxdy = 0.
\end{aligned}
$$

- Since the solution on $\Gamma_D = \partial\Omega/(\Gamma_S \cup \Gamma_R)$ is given by $\mathbf{u} = \mathbf{g}$, then we can choose the test function $\mathbf{v}(x,y)$ such that $\mathbf{v} = 0$ on $\partial\Omega/(\Gamma_S \cup \Gamma_R)$.

# Mixed boundary conditions in normal/tangential directions

- Similar to the derivation of mixed boundary conditions in normal/tangential directions in Chapter 7, we obtain

$$
\begin{aligned}
& \int_{\partial\Omega} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \ ds \\
= & \int_{\Gamma_S} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \ ds + \int_{\Gamma_R} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \ ds \\
& + \int_{\partial\Omega/(\Gamma_S \cup \Gamma_R)} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \ ds \\
= & \left[ \int_{\Gamma_S} p_n(\mathbf{n}^t\mathbf{v}) \ ds + \int_{\Gamma_S} p_\tau(\tau^t\mathbf{v}) \ ds \right] \\
& + \left[ \int_{\Gamma_R} q_n(\mathbf{n}^t\mathbf{v}) \ ds + \int_{\Gamma_R} q_\tau(\tau^t\mathbf{v}) \ ds \right] \\
& - \left[ \int_{\Gamma_R} (r\mathbf{n}^t\mathbf{u})(\mathbf{n}^t\mathbf{v}) \ ds + \int_{\Gamma_R} (r\tau^t\mathbf{u})(\tau^t\mathbf{v}) \ ds \right],
\end{aligned}
$$

# Mixed boundary conditions in normal/tangential directions

- Hence, similar to the treatment of the mixed boundary conditions in normal/tangential directions in Chapter 7, the weak formulation is to find $\mathbf{u} \in H^1(0, T; [H^1(\Omega)]^2)$ and $p \in L^2(0, T; L^2(\Omega))$ such that

$$\begin{aligned}
&\int_\Omega \mathbf{u}_t \cdot \mathbf{v}\, dxdy + \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v}\, dxdy + \int_\Omega 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v})\, dxdy \\
&- \int_\Omega p(\nabla \cdot \mathbf{v})\, dxdy + \int_{\Gamma_R} (r\mathbf{n}^t\mathbf{u})(\mathbf{n}^t\mathbf{v})\, ds + \int_{\Gamma_R} (r\tau^t\mathbf{u})(\tau^t\mathbf{v})\, ds \\
&= \int_\Omega \mathbf{f} \cdot \mathbf{v}\, dxdy + \int_{\Gamma_R} q_n(\mathbf{n}^t\mathbf{v})\, ds + \int_{\Gamma_R} q_\tau(\tau^t\mathbf{v})\, ds \\
&\quad + \int_{\Gamma_S} p_n(\mathbf{n}^t\mathbf{v})\, ds + \int_{\Gamma_S} p_\tau(\tau^t\mathbf{v})\, ds, \\
&- \int_\Omega (\nabla \cdot \mathbf{u})q\, dxdy = 0,
\end{aligned}$$

for any $\mathbf{v} \in [H^1_{0D}(\Omega)]^2$ and $q \in L^2(\Omega)$.

- Code? Combine all of the subroutines for Dirichlet/Stress/Robin boundary conditions.